

Invariant Distributions and local theory of quasiperiodic cocycles in $\mathbb{T}^d \times SU(2)$

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Abstract

We prove local genericity of Distributional Unique Ergodicity (DUE) for certain classes of quasiperiodic cocycles in $\mathbb{T}^d \times SU(2)$, extending and/or refining some preceding results in the field. The proof is based on a more careful analysis of the K.A.M. scheme of [Kri99] and [Kar13], inspired from [Eli02], which also gives a proof of the local density of cocycles which are reducible via finitely differentiable or measurable transfer functions. We then derive some consequences for one-frequency cocycles over recurrent Diophantine rotations thus confirming in this context (and, therefore, in a manifold of dimension 4) a conjecture of A. Katok concerning spaces that carry DUE and cohomologically stable diffeomorphisms.

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1 Introduction

In a recent preprint, [AFK12], A. Avila, B. Fayad and A. Kocsard obtained a class diffeomorphisms serving as counterexamples to a conjecture by G. Forni (see [FFRH13]). This conjecture concerned the spaces that carry *distributionally uniquely ergodic* (DUE) C^∞ -diffeomorphisms, and stated that such spaces, if they are closed manifolds, should be tori. One of the classes of counterexamples constructed was a subset of $SW^\infty(\mathbb{T}, G)$, the space of skew-product diffeomorphisms of $\mathbb{T} \times G$, with G a homogeneous space of the compact type. Our goal is to strengthen their result, more specifically for $G = SU(2)$, on the basis of the recent development of the theory of quasi-periodic cocycles in compact Lie groups by the author in his PhD thesis [Kar13]. We thus obtain a theorem of a different flavour than that of the preprint, where cocycles over a fixed (moreover, Diophantine) frequency are considered. On the other hand, the algebraic setting is more restricted in order to keep the arguments transparent, even though we are confident that the results can be generalized to all semisimple compact Lie groups. We also give a different proof of the theorem of [AFK12] in this setting, using the main theorem of this paper and the theorem on the density of reducible cocycles for a full measure (and thus dense) set of frequencies, proved in [Kri01].

The first step is to recall the part that we need from the theory developed in [Kar13], inspired by [Kri99] and [Eli02], translated in the context of the article. It is the local theory (chapter 8), and more precisely the proof of the theorem on local almost reducibility. Then, we show that if we let the condition devised by H. Eliasson in [Eli02] degenerate in a controlled way, we obtain cocycles that

are reducible via a transfer function in H^σ , for any fixed $\sigma \geq 0$, and not in any H^s , $s > \sigma$.

Theorem 1.1. *Let $\alpha \in DC(\gamma, \tau) \subset \mathbb{T}^d$, and let $\mathcal{W} \subset SW_\alpha^\infty(\mathbb{T}^d, SU(2))$ be the neighborhood of constant cocycles defined in the local almost reducibility theorem (see, e.g. [Kri99] or thm 3.2). Then, for any given $\sigma \in [0, \infty]$, H^σ -reducible cocycles are C^∞ -dense in \mathcal{W} .*

By " H^σ -reducible cocycle", we mean a C^∞ -smooth cocycle which is conjugate to a constant one, and the transfer function is of Sobolev regularity H^σ , but not H^s , for any $s > \sigma$. We remark that the proof implies the same conclusion for finite and negative σ , but we ignore its dynamical content. Finally, since the result is already known for $\sigma = \infty$, see [Kri99], we will focus on the remaining case of finite σ . The whole discussion in paragraphs 6.3 and 6.4 shows the optimality of the K.A.M. scheme for cocycles over a diophantine rotation, satisfying a smallness condition:

Theorem 1.2. *Every cocycle which satisfies the conditions of theorem 3.2 and is reducible in some H^σ , $\sigma \in [0, \infty]$, is "K.A.M.-constructively reducible".*

The word "constructive" is used in a somewhat loose manner, that we explain right-away with the following flowchart, supposing that we have already run the K.A.M. scheme.

- If the product of conjugations converges, it converges in C^∞ and the cocycle is C^∞ -reducible.
- If not, check the summability in H^s of the series δ defined in eq. 9. Then
 - If there exists $s \geq 0$ such that $\delta \in h^s$ and $\sigma \in [0, \infty]$ is the optimal index, the cocycle is H^σ -reducible (and not in higher regularity). Run the algorithm of lemma 6.3 and construct a reducing conjugation. The constant to which the cocycle is reduced is Liouville, by [Kar14].
 - If no such s exists, the cocycle is not reducible.

Subsequently, we use the K.A.M. scheme in order to solve the cohomological equation in $C^\infty(\mathbb{T}^d, SU(2))$, $d \geq 1$, up to an arbitrarily small error, which is equivalent to Distributional Unique Ergodicity of the cocycle. This can be done under a generic condition on the cocycle, constructed after [Eli02], which implies that, for any given function of 0 average, if we go deep enough in the K.A.M. scheme that reduces the given cocycle, the error term in cohomological equation becomes arbitrarily small in the C^∞ topology.

Theorem 1.3. *Let $\alpha \in DC(\gamma, \tau) \subset \mathbb{T}^d$ with $d \in \mathbb{N}^*$. Then, DUE holds in a G_δ dense subset of \mathcal{W} .*

This is a local genericity theorem, and the generic condition that is needed for DUE is strictly stricter than that on [Eli02]. We actually show the "optimality" of the condition, in the sense that if we let it degenerate so that Eliasson's

condition is still satisfied, invariant distributions are created (see prop. 10.1, whose proof is essentially that of thm 1.1 for $\sigma < \infty$).

These theorems hold for cocycles of arbitrary numbers of frequencies. Furthermore, it seems possible that the almost reducibility theorem of [Cha13] can be improved (practically restated) so that it gives an almost reducibility theorem in the sense of definition 3.2 (cf. the comments following the definition). Then, the convergence of the K.A.M. scheme in C_h^ω (i.e. in the analytic category and a fixed width of analyticity) implies a fortiori its convergence in the smooth category, and local genericity of DUE should be true for analytic cocycles, as in the article by H. Eliasson. The following corollary uses the global density theorem 3.1, which has been proved only for one-frequency cocycles ([Kri01],[Fra04],[Kar13]). The density part of the corollary is obtained in the same way as corollary 3.4 from thm 3.1. Since, moreover, DUE is a G_δ condition, we immediately obtain

Corollary 1.4. *If $\alpha \in RDC \subset \mathbb{T}$, DUE is generic in $SW_\alpha^\infty(\mathbb{T}, SU(2))$.*

This was proved in [AFK12] for cocycles over generic Liouvillean rotations, but it is also found under this very restrictive (in the topological sense) arithmetic condition. In order to keep notation and expression simple, we will prove the results only for one-frequency cocycles, for the rest of the article. In the same way we can obtain

Corollary 1.5. *DUE is generic in $SW^\infty(\mathbb{T}, SU(2))$.*

This is, of course, a special case of theorem 4.2. The argument goes like the one in the proof of corollary 5.1.

Proof. DUE is a G_δ property. It holds in a dense set of $SW_\alpha^\infty(\mathbb{T}, SU(2))$, for a dense set of $\alpha \in \mathbb{T}$. \square

In the results above $SW_\alpha^\infty(\mathbb{T}, SU(2))$ can be replaced by $SW_\alpha^{\infty,1}(\mathbb{T}, SO(3))$, the space of cocycles in $\mathbb{T} \times SO(3)$ homotopic to the Id . On the other hand, DUE seems to be a rare property in the remaining connected component of $SW_\alpha^\infty(\mathbb{T}, SO(3))$.

Finally, we address the problem of cohomological stability and prove the following rigidity theorem.

Theorem 1.6. *Under the arithmetic and smallness conditions of theorem 1.3, if a cocycle is cohomologically stable, it is not DUE.*

In fact we prove that cohomological stability implies a very good control of the conjugations produced by the K.A.M. scheme: either the scheme can be made to produce a conjugation, so that the cocycle is actually reducible (to a Diophantine or resonant constant), or the cocycle is UE, and all but finite successive constants in the scheme have orthogonal preimages in $su(2)$. We remind that a diophantine constant $A = \{e^{2i\pi a}, 0\} \in SU(2)$ is a constant for which there exist positive constants λ, σ such that $|2a - k\alpha|_{\mathbb{Z}} \geq \frac{\lambda^{-1}}{|k|^\sigma}$. A constant

cocycle (α, A) is diophantine if α is, and if $A \in G$ is diophantine with respect to α . Since the total space $SW_\alpha^\infty(\mathbb{T}, SU(2))$, $\alpha \in RDC$, is filled by cocycles conjugate to diagonal ones (see section 11), we obtain the following theorem.

Theorem 1.7. *If $\alpha \in RDC$, there are no counterexamples to conjecture 4.1 in $SW_\alpha^\infty(\mathbb{T}, SU(2))$.*

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2 Facts from algebra and arithmetics

2.1 The group $SU(2)$

The matrix group $SU(2)$ is the multiplicative group of unitary 2×2 matrices of determinant 1. For the greatest part of the article, unless the contrary is explicitly stated, $G = SU(2)$.

Let us denote the matrix $S \in G$, $S = \begin{pmatrix} z & w \\ -\bar{z} & \bar{w} \end{pmatrix}$, where $(z, w) \in \mathbb{C}^2$ and $|z|^2 + |w|^2 = 1$, by $\{z, w\}_G$. The subscript will be suppressed from the notation, unless necessary. The manifold $G = SU(2)$ is thus naturally identified with $\mathbb{S}^3 \subset \mathbb{C}^2$, and, in particular, G is simply connected. The law of multiplication is pushed forward to the mapping of $\mathbb{S}^3 \times \mathbb{S}^3 \rightarrow \mathbb{S}^3$ given by $\{z_1, w_1\} \cdot \{z_2, w_2\} = \{z_1 z_2 - w_1 \bar{w}_2, z_1 w_2 + \bar{z}_2 w_1\}$. Inversion is pushed forward to the involution of \mathbb{S}^3 , $\{z, w\} \mapsto \{\bar{z}, -w\}$. When coordinates in \mathbb{C}^2 are fixed, the circle \mathbb{S}^1 is naturally embedded in G as the group of diagonal matrices, which is a maximal torus (i.e. a maximal abelian subgroup) of G .

The Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$ is naturally isomorphic to $\mathbb{R}^3 \approx \mathbb{R} \times \mathbb{C}$ equipped with its vector and scalar product. It will be denoted by \mathfrak{g} . The element $s = \begin{bmatrix} it & u \\ -\bar{u} & -it \end{bmatrix}$ will be denoted by $\{t, u\}_\mathfrak{g} \in \mathbb{R} \times \mathbb{C}$. The scalar product is defined by

$$\langle \{t_1, u_1\}, \{t_2, u_2\} \rangle = t_1 t_2 + \mathcal{R}(u_1 \bar{u}_2) = t_1 t_2 + \mathcal{R}u_1 \mathcal{R}u_2 + \mathcal{I}u_1 \mathcal{I}u_2$$

Mappings with values in $\mathfrak{su}(2)$ will be denoted by

$$U(\cdot) = \{U_t(\cdot), U_z(\cdot)\}_\mathfrak{g}$$

in these coordinates, where $U_t(\cdot)$ is a real-valued and $U_z(\cdot)$ is a complex-valued function.

The adjoint action of $h \in \mathfrak{su}(2)$ on itself is pushed-forward to twice the vector product:

$$ad_{\{1,0\}} \cdot \{0,1\} = [\{1,0\}, \{0,1\}] = 2\{0,i\}$$

plus cyclic permutations, and the Cartan-Killing form, normalized by $\langle h, h' \rangle = -\frac{1}{8\pi} \text{tr}(ad(h) \circ ad(h'))$ is pushed-forward to the scalar product of \mathbb{R}^3 . The periodic geodesics of the group for the induced Riemannian structure are of the form $S.E_r(\cdot).S^{-1} = S.\exp(\{2\pi r \cdot, 0\}_g).S^{-1}$, $S \in G$ and $r \in \mathbb{N}^*$.

Under this normalization, the minimal length for a geodesic leaving from the Id and arriving at $-Id$ is 1, and the minimal length of a periodic geodesic is twice as much. We also find directly that the preimages of the Id in the maximal toral (i.e. abelian) algebra of diagonal matrices are points of coordinates in the lattice $2\pi\mathbb{Z}$.

The adjoint action of the group on its algebra is pushed-forward to the action of $SO(3)$ on $\mathbb{R} \times \mathbb{C}$. In particular, the diagonal matrices, of the form $S = \exp(\{2\pi s, 0\}_g)$, $Ad(S).\{t, u\} = \{t, e^{4i\pi s}u\}$.

Finally, the Weyl group of $SU(2)$ (i.e. the group of symmetries of $\mathcal{T} \hookrightarrow SU(2)$ induced by $SU(2)$) is naturally isomorphic to the multiplicative group $\{1, -1\}$ and two representatives are the matrices

$$Id \text{ and } \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

the action of the second one being a reflection with respect to the complex plane:

$$(-1)_W.\{t, 0\}_{su(2)} = \{-t, 0\}_{su(2)}$$

The normalizer \mathcal{N} of the subgroup \mathcal{T} of diagonal matrices contains \mathcal{T} itself, and matrices of the form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.\mathcal{T}$. The diagonalization of matrices in G uses elements of $G \bmod \mathcal{N}$.

2.2 Functional Spaces

We will consider the space $C^\infty(\mathbb{T}, g)$ equipped with the standard maximum norms

$$\|U\|_s = \max_{0 \leq \sigma \leq s} \max_{\mathbb{T}} |\partial^\sigma U(\cdot)|$$

for $s \geq 0$, and the Sobolev norms

$$\|U\|_{H^s}^2 = \sum_{k \in \mathbb{Z}^d} (1 + |k|^2)^s |\hat{U}(k)|^2$$

where $\hat{U}(k) = \int U(\cdot) e^{-2i\pi kx}$ are the Fourier coefficients of $U(\cdot)$. The fact that the injections $H^{s+1/2}(\mathbb{T}, g) \hookrightarrow C^s(\mathbb{T}, g)$ and $C^s(\mathbb{T}, g) \hookrightarrow H^s(\mathbb{T}, g)$ for all $s \geq 0$ are continuous is classical.

By abusing the notation, we note $H^0 = L^2$, and define the (not complete) spaces

$$\tilde{H}^\sigma = H^\sigma \setminus \{\cup_{s>\sigma} H^s\}, \sigma \geq 0$$

The spaces with tildes are generic subspaces of the corresponding Hilbert spaces, and consist of the elements that exhibit no additional Sobolev regularity. We

will denote the corresponding spaces of complex sequences by lowercase letters,

$$\begin{aligned} h^s &= \{f \in \ell^2, \sum (1+n)^{2s} |f_n|^2 < \infty\} \\ \tilde{h}^\sigma &= h^\sigma \setminus \{\cup_{s>\sigma} h^s\} \end{aligned}$$

For this part, see [Fol95] and [SW71]. In view of the identification of the C^∞ manifold $G = SU(2)$ with $\mathbb{S}^3 \subset \mathbb{C}^2$, equipped with the natural measure normalized so that sphere has volume 1, the space $C^\infty(G)$ of smooth \mathbb{C} -valued functions defined on G , can be identified with $C^\infty(\mathbb{S}^3)$, and the identification is an isometry between the L^2 spaces. In order to obtain a basis facilitating the calculations, we use the representation theory of G . Given a system of coordinates (ζ, ω) in \mathbb{C}^2 , we can define an orthonormal basis for \mathcal{P}_m , the space of homogenous polynomials of degree m , by $\{\psi_{l,m}\}_{0 \leq l \leq m}$ where $\psi_{l,m}(\zeta, \omega) = \sqrt{\frac{(m+1)!}{l!(m-l)!}} \zeta^l \omega^{m-l}$. The group G acts on \mathcal{P}_m by $\{z, w\} \cdot \phi(\zeta, \omega) = \phi(z\zeta + w\omega, -\bar{w}\zeta + \bar{z}\omega)$, and the resulting representation is noted by π_m . For m fixed, we can define the matrix coefficients relative to the basis by $\pi_m^{j,p}\{z, \bar{z}, w, \bar{w}\} \mapsto \langle \{z, w\} \cdot \psi_{j,m}, \psi_{p,m} \rangle$. The matrix coefficients are harmonic functions of z, \bar{z}, w, \bar{w} , and are of bidegree $(m-p, p)$, i.e. they are homogeneous of degree $m-p$ in (z, w) , and homogeneous of degree p in (\bar{z}, \bar{w}) , and they generate the space \mathcal{E}_{π_m} .

We thus obtain the decomposition $L^2 = \oplus_{m \in \mathbb{N}} \mathcal{E}_{\pi_m} = \oplus_{m \in \mathbb{N}} \oplus_{0 \leq p \leq m} \mathcal{H}_{m,p}$, where $\mathcal{H}_{m,p}$ is spanned by $\{\pi_m^{j,p}, 0 \leq j \leq m\}$. In this decomposition, the harmonic functions are regrouped according to their bidegree.

Therefore, given a system of coordinates in \mathbb{C}^2 , a function $f \in L^2(\mathbb{S}^3)$ can be written in the form

$$f(z, \bar{z}, w, \bar{w}) = \sum_{m \in \mathbb{N}} \sum_{0 \leq p \leq m} \sum_{0 \leq j \leq m} f_{j,p}^m \pi_m^{j,p}(z, \bar{z}, w, \bar{w})$$

where $f_{j,p}^m \in \mathbb{C}$ are the Fourier coefficients. The functions $\pi_m^{j,p}(z, \bar{z}, w, \bar{w})$ are the eigenvectors of the Laplacian on \mathbb{S}^3 and consequently smooth (in fact real analytic), and they form an orthonormal basis for $L^2(\mathbb{S}^3)$. In higher regularity, they generate a dense subspace of C^∞ .

The group G acts on $C^\infty(G) \equiv C^\infty(\mathbb{S}^3)$ by pullback: if $A \in G$ and $\begin{pmatrix} z \\ w \end{pmatrix} \in \mathbb{S}^3$, then, for $\phi : \mathbb{S}^3 \rightarrow \mathbb{C}$,

$$(A \cdot \phi) \left(\begin{pmatrix} z \\ w \end{pmatrix} \right) = \phi \left(A^* \begin{pmatrix} z \\ w \end{pmatrix} \right)$$

If coordinates are chosen so that $A = \{e^{2i\pi a}, 0\}$ is diagonal, then

$$A \cdot \phi(z, w) = \phi(e^{-2i\pi a} z, e^{2i\pi a} w)$$

and A then acts on harmonic functions by

$$A \cdot \pi_m^{j,p}(z, \bar{z}, w, \bar{w}) = e^{2i\pi(m-2p)a} \pi_m^{j,p}(z, \bar{z}, w, \bar{w})$$

where, of course, $m - 2p = m - p - p$ is the difference of the degree of homogeneity in (\bar{z}, \bar{w}) and (z, w) . Therefore, the harmonics in these coordinates are eigenvectors for the associated operator. In particular, if a is irrational, the eigenvectors for the eigenvalue 1 are exactly the elements $\pi_m^{j, m/2}$, $0 \leq j \leq m$.

The group of symmetries of $\mathbb{C}.\psi_{m/2, m}$ is exactly the normalizer \mathcal{N} of \mathcal{T} . The subspaces $\mathbb{C}.\psi_{l, m}$, $l \neq m/2$ appear to have a smaller normalizer, \mathcal{T} , but the normalizer of $\mathbb{C}.\psi_{l, m} + \mathbb{C}.\psi_{m-l, m}$ is in deed \mathcal{N} . The following two crucial lemmata concern the effect of a change of basis in \mathbb{C}^2 on the Fourier development of f and are proofs and quantifications of this fact.

Lemma 2.1. *Let $\psi_{l, m} \in \mathcal{P}_m$. Let also $\pi_{p, m}$ the projection on $\mathbb{C}.\psi_{p, m}(\zeta, \omega)$. Then, unless $l' \in \{l, m-l\}$, there exists a constant $C_{l, m} < 1$ such that for every $A \in G$*

$$\|\pi_{l', m}(A.\psi_{l, m}(\zeta, \omega))\| \leq C_{l, m}$$

In fact we show that the only possible way of transforming a monomial in \mathcal{P}_m to another one by a linear change of coordinates is by permuting the coordinates (and changing the argument within each complex plane), so that $\mathbb{C}.\psi_{l, m}(\zeta, \omega) + \mathbb{C}.\psi_{m-l, m}(\zeta, \omega)$ is preserved exactly by \mathcal{N} .

Proof. Elementary calculation shows that

$$|\psi_{l, m}(\zeta, \omega)| \leq \sqrt{\frac{(m+1)!}{l!(m-l)!}} \left(\frac{l}{m}\right)^{l/2} \left(\frac{m-l}{m}\right)^{(m-l)/2}$$

and, if $l < m$, the locus where the maximum is attained is

$$\left\{ \left(\sqrt{\frac{l}{m}} e^{2i\pi\theta}, \sqrt{\frac{m-l}{m}} e^{-2i\pi \frac{m-l}{m-l} \theta} \right) \right\}_{\theta \in \mathbb{T}}$$

If $l = m$, the maximum is attained at $\{(e^{2i\pi\theta}, 0)\}_{\theta \in \mathbb{T}}$. Clearly, the loci corresponding to harmonics not obtained by permutations of coordinates cannot be obtained by linear transformations of the sphere, and therefore the only vector in the standard basis of \mathcal{P}_m that can be obtained by the action of G on $\psi_{l, m}$ is $\psi_{m-l, m}(\zeta, \omega)$. In particular, the only transformations leaving $\mathbb{C}.\psi_{m/2, m}(\zeta, \omega)$ fixed are

$$\begin{pmatrix} e^{2i\pi\theta} & 0 \\ 0 & e^{-2i\pi\theta} \end{pmatrix} \text{ and } \begin{pmatrix} 0 & e^{2i\pi\theta} \\ -e^{-2i\pi\theta} & 0 \end{pmatrix}$$

for $\theta \in \mathbb{T}$. □

The second lemma examines the effect of changes of coordinates on the eigenvectors for the eigenvalue 1, $\psi_{m/2, m}$.

Lemma 2.2. *For a given $m > 0$ and even, there exist m points $D \in G/\mathcal{N}$ such that $\pi_{m/2, m}(D.\psi_{m/2, m}) = 0$.*

Proof. Call $l = m/2$. We can calculate explicitly the projection:

$$\pi_{l,m}(\{z, w\}_G \cdot \psi_{l,m}) = \sum_0^l (-1)^i \binom{l}{i}^2 |z|^{2(l-i)} |w|^{2i} \psi_{l,m} = p_l(|z|, |w|) \psi_{l,m}$$

The factor of the projection, p_l , is a Legendre polynomial of the variable $|z|^2$ and $|w|^2 = 1 - |z|^2$. The fact that all its roots are real and in the interval $[0, 1]$ is classical, see for example [Arn04]. \square

Returning to more general facts from calculus, the C^s norms for functions in $C^\infty(G)$ are defined in a classical way, and the Sobolev norms are defined by

$$\|f\|_{H^s}^2 = \sum_{m,j,p} (1 + m^2)^s |f_m^{j,p}|^2$$

where $f_m^{j,p}$ are the coefficients of the harmonics in the expansion of f .

We will also use the convexity or Hadamard-Kolmogorov inequalities (see [Kol49]) ($U \in C^\infty(\mathbb{T}, g)$):

$$\|U(\cdot)\|_\sigma \leq C_{s,\sigma} \|U\|_0^{1-\sigma/s} \|U\|_s^{\sigma/s} \quad (1)$$

for $0 \leq \sigma \leq s$, and the inequalities concerning the composition of functions (see [Kri99]):

$$\|\phi \circ (f + u) - \phi \circ f\|_s \leq C_s \|\phi\|_{s+1} (1 + \|f\|_0)^s (1 + \|f\|_s) \|u\|_s \quad (2)$$

We will use the truncation operators for mappings $\mathbb{T} \rightarrow g$ defined by

$$\begin{aligned} T_N f(\cdot) &= \sum_{|k| \leq N} \hat{f}(k) e^{2i\pi k \cdot} \\ \dot{T}_N f(\cdot) &= T_N f(\cdot) - \hat{f}(0) \\ R_N f(\cdot) &= \sum_{|k| > N} \hat{f}(k) e^{2i\pi k \cdot} \text{ and} \\ \dot{T} f(\cdot) &= T f(\cdot) - \hat{f}(0) \end{aligned}$$

These operators satisfy the estimates

$$\|T_N f(\cdot)\|_{C^s} \leq C_s N \|f(\cdot)\|_{C^s} \quad (3)$$

$$\|R_N f(\cdot)\|_{C^s} \leq C_{s,s'} N^{s-s'+2} \|f(\cdot)\|_{C^{s'}} \quad (4)$$

The Fourier spectrum of a function will be denoted by $\hat{\sigma}(f) = \{k \in \mathbb{Z}, \hat{f}(k) \neq 0\}$.

Finally, for functions $f : G \rightarrow \mathbb{C}$, we truncate on m :

$$T_M f(\cdot) = \sum_{m \leq M} \sum_{0 \leq j,p \leq m} f_{j,p}^m \pi_m^{j,p}$$

and likewise for the other operators. These operators satisfy the estimates

$$\|T_M f(\cdot)\|_{C^s} \leq C_s M^2 \|f(\cdot)\|_{C^s}$$

$$\|R_M f(\cdot)\|_{C^s} \leq C_{s,s'} M^{s-s'+2} \|f(\cdot)\|_{C^{s'}}$$

2.3 Arithmetics, continued fraction expansion

For this section, see [Khi63] or [Arn83]. Let us introduce some notation. For $\alpha \in \mathbb{R}^*$, define

1. $|\alpha|_{\mathbb{Z}} = \text{dist}(\alpha, \mathbb{Z}) = \min_{\mathbb{Z}} |\alpha - l|$
2. $[\alpha]$ the integer part of α
3. $\{\alpha\}$ the fractional part of α
4. $G(\alpha) = \{\alpha^{-1}\}$, the Gauss map

Consider $\alpha \in \mathbb{T} \setminus \mathbb{Q}$ fixed, and let $p_{-2} = q_{-1} = 0$ and $p_{-1} = q_{-2} = 1$. Then $(p_n/q_n)_{n \geq 0}$ is the sequence of best rational approximations of α defined recursively as follows. Let $\alpha_n = G^n(\alpha) = G(\alpha_{n-1})$, $a_n = [\alpha_{n-1}^{-1}]$, and $\beta_n = \prod_{k=0}^n \alpha_k$. Then the Euclidean division of β_{n-2} by β_{n-1} reads

$$\beta_{n-2} = a_n \beta_{n-1} + \beta_n$$

and we can define

$$\begin{aligned} q_n &= a_n q_{n-1} + q_{n-2} \\ p_n &= a_n p_{n-1} + p_{n-2} \end{aligned}$$

p_n and q_n are strictly positive for $n \geq 1$ and $\beta_n = (-1)^n (q_n \alpha - p_n)$. We have

$$\begin{aligned} \frac{1}{q_n + q_{n+1}} &< \beta_n < \frac{1}{q_{n+1}} \\ |q_{n-1} \alpha|_{\mathbb{Z}} &< |k \alpha|_{\mathbb{Z}}, \quad \forall 0 < k < q_n \end{aligned}$$

The following notion is essential in K.A.M. theory. It is related with the quantification of the closeness of rational numbers to certain classes of irrational numbers.

Definition 2.1. We will denote by $DC(\gamma, \tau)$ the set of numbers α in $\mathbb{T} \setminus \mathbb{Q}$ such that for any $k \neq 0$, $|\alpha k|_{\mathbb{Z}} \geq \frac{\gamma^{-1}}{|k|^\tau}$. Such numbers are called *Diophantine*.

The set $DC(\gamma, \tau)$, for $\tau > 2$ fixed and $\gamma \in \mathbb{R}_+^*$ small is of positive Haar measure in \mathbb{T} . If we fix τ and let γ run through the positive real numbers, we obtain $\cup_{\gamma > 0} DC(\gamma, \tau)$ which is of full Haar measure. The numbers that do not satisfy any Diophantine condition are called *Liouvillean*. They form a residual set of 0 Lebesgue measure.

This last following definition concerns the relation of the approximation of an irrational number with its continued fractions representation.

Definition 2.2. We will denote by $RDC(\gamma, \tau)$ the set of recurrent Diophantine numbers, i.e. the α in $\mathbb{T} \setminus \mathbb{Q}$ such that $G^n(\alpha) \in DC(\gamma, \tau)$ for infinitely many n .

The set RDC is also of full measure, since the Gauss map is ergodic with respect to a smooth measure. In contexts where the parameters γ and τ are not significant, they will be omitted in the notation of both sets.

3 Cocycles in $\mathbb{T}^d \times SU(2)$

3.1 Definition of the dynamics

Let $\alpha \in \mathbb{T}^d \equiv \mathbb{R}^d / \mathbb{Z}^d$, $d \in \mathbb{N}^*$, be an irrational rotation, so that the translation $x \mapsto x + \alpha \pmod{\mathbb{Z}^d}$ is minimal and uniquely ergodic. For the greatest part of the article $d = 1$, and the results holding for cocycles over rotations in tori of higher dimension will be proved for $d = 1$, but stated in full generality.

If we also let $A(\cdot) \in C^\infty(\mathbb{T}^d, G)$, the couple $(\alpha, A(\cdot))$ acts on the fibered space $\mathbb{T}^d \times G \rightarrow \mathbb{T}^d$ defining a diffeomorphism by

$$(\alpha, A(\cdot)).(x, S) = (x + \alpha, A(x).S)$$

for any $(x, S) \in \mathbb{T}^d \times G$. We will call such an action a *quasiperiodic cocycle over R_α* (or simply a cocycle). The space of such actions is denoted by $SW_\alpha^\infty(\mathbb{T}^d, G) \subset Diff^\infty(\mathbb{T}^d \times G)$. Most times we will abbreviate the notation to SW_α^∞ . Cocycles are a class of fibered diffeomorphisms, since fibers of $\mathbb{T}^d \times G$ are mapped into fibers, and the mapping from one fiber to another in general depends on the base point. The number $d \in \mathbb{N}^*$ is the number of frequencies of the cocycle.

The space $\bigcup_{\alpha \in \mathbb{T}^d} SW_\alpha^\infty$ will be denoted by SW^∞ . The space SW_α^∞ inherits the topology of $C^\infty(\mathbb{T}^d, G)$, and SW^∞ has the standard product topology of $\mathbb{T}^d \times C^\infty(\mathbb{T}^d, G)$. We note that cocycles are defined over more general maps and in more general contexts of regularity and structure of the basis and fibers.

If we consider a representation of G on a vector space E , the action of the cocycle can be also defined on $\mathbb{T}^d \times E$, simply by replacing S by a vector in E and multiplication in G by the action. The particular case which will be important in this article is the representation of G on $L^2(G)$, and the resulting action of the cocycle on $L^2(\mathbb{T}^d \times G)$.

The n -th iterate of the action is given by

$$(\alpha, A(\cdot))^n.(x, S) = (n\alpha, A_n(\cdot)).(x, S) = (x + n\alpha, A_n(x).S)$$

where $A_n(\cdot)$ represents the *quasiperiodic product* of matrices equal to

$$A_n(\cdot) = A(\cdot + (n-1)\alpha) \dots A(\cdot)$$

for positive iterates. Negative iterates are found as inverses of positive ones:

$$(\alpha, A(\cdot))^{-n} = ((\alpha, A(\cdot))^n)^{-1} = (-n\alpha, A^*(\cdot - n\alpha) \dots A^*(\cdot - \alpha))$$

3.2 Classes of cocycles with simple dynamics, conjugation

The cocycle $(\alpha, A(\cdot))$ is called a constant cocycle if $A(\cdot) = A \in G$ is a constant mapping. In that case, the quasiperiodic product reduces to a simple product of matrices

$$(\alpha, A)^n = (n\alpha, A^n)$$

The group $C^\infty(\mathbb{T}^d, G) \hookrightarrow SW^\infty(\mathbb{T}^d, G)$ acts by *fibered conjugation*: Let $B(\cdot) \in C^\infty(\mathbb{T}^d, G)$ and $(\alpha, A(\cdot)) \in SW^\infty(\mathbb{T}^d, G)$. Then we define

$$\begin{aligned} Conj_{B(\cdot)}(\alpha, A(\cdot)) &= (0, B(\cdot)) \circ (\alpha, A(\cdot)) \circ (0, B(\cdot))^{-1} \\ &= (\alpha, B(\cdot + \alpha).A(\cdot).B^{-1}(\cdot)) \end{aligned}$$

which is in fact a change of variables within each fiber of the product $\mathbb{T}^d \times G$. The dynamics of $Conj_{B(\cdot)}(\alpha, A(\cdot))$ and $(\alpha, A(\cdot))$ are essentially the same, since

$$(Conj_{B(\cdot)}(\alpha, A(\cdot)))^n = (n\alpha, B(\cdot + n\alpha).A_n(\cdot).B^{-1}(\cdot))$$

Definition 3.1. Two cocycles $(\alpha, A(\cdot))$ and $(\alpha, \tilde{A}(\cdot))$ in SW^∞_α are conjugate iff there exists $B(\cdot) \in C^\infty(\mathbb{T}^d, G)$ such that $(\alpha, \tilde{A}(\cdot)) = Conj_{B(\cdot)}(\alpha, A(\cdot))$. We will use the notation

$$(\alpha, A(\cdot)) \sim (\alpha, \tilde{A}(\cdot))$$

to state that the two cocycles are conjugate to each other.

Since constant cocycles are a class for which dynamics can be analysed, we give the following definition.

Definition 3.2. A cocycle will be called *reducible* iff it is conjugate to a constant.

Due to the fact that not all cocycles are reducible (e.g. generic cocycles in $\mathbb{T} \times \mathbb{S}^1$ over Liouvillean rotations, but also cocycles over Diophantine rotations, even though this result is hard to obtain, see [Eli02], [Kri01]) we also need the following concept, which has proved to be crucial in the study of such dynamical systems.

Definition 3.3. A cocycle $(\alpha, A(\cdot))$ is said to be *almost reducible* if there exists a sequence of conjugations $B_n(\cdot) \in C^\infty$, such that $Conj_{B_n(\cdot)}(\alpha, A(\cdot))$ becomes arbitrarily close to constants in the C^∞ topology, i.e. iff there exists (A_n) , a sequence in G , such that

$$A_n^*(B_n(\cdot + \alpha)A(\cdot)B_n^*(\cdot)) \xrightarrow{C^\infty} Id$$

Usually, this property is established in a K.A.M. constructive way, making it possible to measure the rate of convergence versus the explosion of the conjugations. Even though it is not a part of the definition, if we call $F_n(\cdot) \in C^\infty(\mathbb{T}^d, g)$ the error term which makes this last limit into an equality, almost reducibility then comes along with obtaining that

$$Ad(B_n(\cdot)).F_n(\cdot) = B_n(\cdot).F_n(\cdot).B_n^*(\cdot) \xrightarrow{C^\infty} 0$$

If this additional condition is satisfied, almost reducibility in the sense of the definition above and almost reducibility in the sense that "the cocycle can be conjugated arbitrarily close to reducible cocycles" are equivalent.

We can now recall the main theorem of [Kar13].

Theorem 3.1. *Let $\alpha \in RDC \subset \mathbb{T}$ and G a semisimple compact Lie group, and call $SW_\alpha^{\infty,1}(\mathbb{T}, G)$ the space of cocycles over α which are homotopic to the Id . Then, reducible cocycles are dense in $SW_\alpha^{\infty,1}(\mathbb{T}, G)$ in the C^∞ topology.*

The following theorem is used the proof of the local density theorem, already proved in [Kri99]. We will sketch the proof of [Kar13], since it implies a useful corollary (cor. 6.2) in a more direct manner than the previously existing proofs.

Theorem 3.2. *Let $\alpha \in DC(\gamma, \tau) \subset \mathbb{T}^d, d \geq 1$ and G a semisimple compact Lie group. Then, there exists $s_0 \in \mathbb{N}^*$ and $\epsilon > 0$, such that if $(\alpha, Ae^{F(\cdot)}) \in SW_\alpha^\infty(\mathbb{T}^d, G)$ with $\|F(\cdot)\|_0 < \epsilon$ and $\|F(\cdot)\|_{s_0} < 1$, $(\alpha, Ae^{F(\cdot)})$ is almost reducible. The neighborhood of constants thus defined will be denoted by $\mathcal{W} = \mathcal{W}_{\alpha,d,G}$.*

As a continuation of the proof of this theorem for cocycles in $\mathbb{T}^d \times SO(3)$, H. Eliasson, in his article cited above, proved the following one. Originally, the theorem was proved for analytic cocycles, which is a stronger result, but the proof works equally well in the smooth category¹.

Theorem 3.3 (H. Eliasson). *Let $G = SU(2)$. Then, unique ergodicity is generic in the set of cocycles satisfying the conditions of theorem 3.2.*

We can also prove the following corollary.

Corollary 3.4. *If $\alpha \in RDC$ and $G = SU(2), SO(3)$, uniquely ergodic (UE) cocycles are generic in $SW_\alpha^{\infty,1}(\mathbb{T}, G)$ in the C^∞ topology.*

Proof. Any given cocycle homotopic to the Id is accumulated by reducible cocycles, and reducible cocycles are accumulated by UE cocycles. Unique ergodicity is a G_δ condition in the C^0 topology. \square

Finally, for completeness, we give the following theorem which belongs to the non-local regime of the dynamics (see [Kri01], [Fra04], [Kar13]), which is an intermediate result in the proof of thm 3.1 for $G = SU(2)$.

Theorem 3.5. *Let $G = SU(2)$ and $\alpha \in RDC$. Then, all cocycles which are not almost-reducible, are conjugate to a diagonal cocycle of the form $(\alpha, E_r(\cdot))$, where $E_r(\cdot) = \{e^{2i\pi r \cdot}, 0\}$ and $r \in \mathbb{N}^*$.*

4 Invariant Distributions and DUE

If we let M be a compact C^∞ -manifold without boundary, $C^\infty(M) = C^\infty(M, \mathbb{C})$, the space of C^∞ -functions on M , is a Fréchet space for the C^∞ topology. Every $f \in Diff^\infty(M)$ acts on $C^\infty(M)$ by pullback: for $\phi \in C^\infty(M)$ we define

$$f^*\phi(\cdot) = \phi \circ f(\cdot)$$

¹The genericity of H. Eliasson's condition in C^∞ will be proved in section 6.

For a fixed f this action is clearly linear and continuous. Therefore, f acts by the transposed action on the dual space of $C^\infty(M)$, the space of distributions $\mathcal{D}'(M)$. This action is given by

$$\langle f_*T, \phi \rangle = \langle T, f^*\phi \rangle$$

for all $T \in \mathcal{D}'(M)$ and for all $\phi \in C^\infty(M)$. We also remind that for any given distribution T , there exists a minimal $k \in \mathbb{N}$, called the order of T , such that T extends to a continuous functional on $C^k(M)$ by density of $C^\infty(M) \hookrightarrow C^k(M)$. Distributions of order 0 are simply the (signed) measures of finite mass on M . Therefore, for any given $f \in \text{Diff}^\infty(M)$ we can define the space of f -invariant distributions as

$$\mathcal{D}'_f(M) = \{T \in \mathcal{D}'(M), f_*T = T\}$$

This space is non-empty, since it contains the (vector space generated by the) invariant measures of f , so it has dimension at least 1.

Definition 4.1. *A diffeomorphism f is DUE iff $\mathcal{D}'_f(M)$ is one-dimensional.*

In particular, Distributional Unique Ergodicity implies Unique Ergodicity. The property of being DUE, however, refines UE, since a space much larger than the space of Radon measures, that of distributions, is considered.

We will use an alternative definition of DUE, stated in terms of the cohomological equation. It is a direct consequence of the Hahn-Banach theorem. The space $C_0^\infty(M)$ is the space of 0-mean-value functions on M for the unique ergodic measure of f .

Lemma 4.1. *A diffeomorphism $f \in \text{Diff}^\infty(M)$ is DUE iff for every $\phi(\cdot) \in C_0^\infty(M)$ and every $\epsilon, s_0 > 0$, there exist $\varepsilon(\cdot) \in C^\infty(M)$, satisfying $\|\varepsilon(\cdot)\|_{s_0} \leq \epsilon$, and $\psi \in C^\infty(M)$ such that*

$$\phi(\cdot) = f^*\psi(\cdot) - \psi(\cdot) + \varepsilon(\cdot)$$

In other words, f is DUE iff *coboundaries* (i.e functions of the form $f^*\psi' - \psi'$ for some $\psi' \in C^\infty$) are dense in $C_0^\infty(M)$. If the space of coboundaries is closed, then f is called *cohomologically stable*. It is a classical fact that Diophantine rotations in tori are both DUE and cohomologically stable, while Liouvillean rotations are only DUE. The following conjecture concerns the converse (see [Hur85], problem 17).

Conjecture 4.1 (A. Katok). *If $\varphi \in \text{Diff}^\infty(M)$ is DUE and cohomologically stable, then it is smoothly conjugate to a Diophantine rotation in a torus \mathbb{T}^d .*

This conjecture has only been proved partially, see [Koc09], [Koc07] and references therein. Theorem 1.3 does not provide counterexamples to the conjecture (thm 1.7). The reason why cocycles are DUE is that the dynamics in the fibers look like a Liouvillean rotation around a fixed axis for some time long enough for equidistribution of orbits in each leaf $\approx \mathbb{T} \times \mathbb{S}^1$ to be established to a certain extent. Then, the axis tilts, and the procedure continues for an infinite

number of times. The Liouvillean character of the dynamics rules out cohomological stability, and an additional generic condition on the angles between the successive constants, stricter than the one needed for UE, is needed in order to establish DUE.

We note that it is possible that f be uniquely ergodic, but $\dim(\mathcal{D}'_f(M)) > 1$, see [Kat01] for the study of the parabolic map

$$\begin{aligned} \mathbb{T} \times \mathbb{S}^1 &\rightarrow \mathbb{T} \times \mathbb{S}^1 \\ (x, e^{2i\pi\theta}) &\mapsto (x + \alpha, e^{2i\pi(\theta+x)}) \end{aligned}$$

This map corresponds to the models of positive energy cocycles, which are the obstructions to almost reducibility in $SW^\infty_\alpha(\mathbb{T}, SU(2))$, $\alpha \in RDC$ (cf. thm 3.5). Its invariant distributions, which can be written explicitly in terms of their Fourier expansion, correspond to the obstructions perturbations which diminish the degree of the cocycle. For the same reason, they manifest themselves as obstructions to conjugation to the exact model in its local theory. Moreover, the fact that all invariant distributions are in $H^{-1}(\mathbb{T} \times SU(2))$ provides an explanation for the fact that the degree is defined for H^1 cocycles, but does not seem to admit a generalization to less regular cocycles.

The part of the theorem of [AFK12] which we are interested in reads

Theorem 4.2 (A.Avila-B.Fayad-A.Kocsard). *DUE diffeomorphisms are generic in $\mathcal{AK}^\infty \subset Diff^\infty(\mathbb{T} \times P)$ defined as the closure of*

$$\{Conj_{B(\cdot)}(\alpha, Id), \alpha \in \mathbb{T}, B(\cdot) \in C^\infty(\mathbb{T}, P)\} \quad (5)$$

in $Diff^\infty(\mathbb{T} \times P)$. Here, $P = G/H$ with G a compact Lie group and H a closed subgroup.

We will also use the following fact, proved in the same reference

Lemma 4.3. *The set of DUE diffeomorphisms of $\mathbb{T} \times P$, P as in the theorem, is a G_δ subset of $Diff^\infty$.*

In this context G is not necessarily semi-simple as in [Kar13]. Spaces defined as P are called homogeneous spaces of compact type. We note that the space of DUE diffeomorphisms constructed in the proof of the theorem is a subspace of $SW^\infty(\mathbb{T}, G)$ acting on $\mathbb{T} \times P$, where G acts on P on the left.

5 Some direct consequences

The proof of thm 4.2, combined with thm 3.1, gives the following corollary which identifies the space where DUE lives as a big space, showing that DUE is not an exotic property in these non-abelian extensions of rotations.

Corollary 5.1. *DUE is a G_δ -dense property in $SW^{\infty,1}(\mathbb{T}, G)$, where G is a semisimple compact Lie group.*

This is the only part of the article where we consider Lie groups other than $SU(2)$, and a reader unfamiliar with the theory of semi-simple compact Lie groups can replace G with $SU(N)$.

Proof. We begin by observing that theorem 3.1, together with the density of RDC in \mathbb{T} , implies that reducible cocycles are dense in $SW^{\infty,1}(\mathbb{T}, G)$, since we allow the frequency to vary. Secondly, we point out that the proof of thm 4.2 actually shows that DUE is G_δ -dense in the Anosov-Katok space

$$\mathcal{AK}^\infty = cl^\infty\{Conj_{B(\cdot)}(\alpha, Id), \alpha \in RDC, B(\cdot) \in C^\infty(\mathbb{T}, G)\}$$

The only part missing in the proof is actually showing that this space coincides with the space of cocycles with values in G , which are homotopic to the Id .

In what follows, we drop the rigor in order to skip some technicalities on semisimple compact Lie groups. Let us fix an embedding $\mathbb{T}^w \equiv \mathcal{T} \subset G$ such that $w \in \mathbb{N}^*$ is maximal. Therefore, we also fix \mathcal{Z} , the lattice of preimages of $Id \in G$ under the \exp in $\mathbb{R}^w \equiv T_0\mathbb{T}^w$. Then, we see that reducibility of a cocycle to (α, Id) with $\alpha \in \mathbb{T} \setminus \mathbb{Q}$ implies reducibility to $(\alpha, \exp(s))$, where $s \in \mathbb{R}^w$ satisfies $s \in \alpha\mathcal{Z}$. This can be seen by considering

$$\{Conj_{B(\cdot)}(\alpha, Id), B(\cdot) : \mathbb{T} \rightarrow \mathcal{T}, \text{ a morphism}\}$$

By construction, $B(\alpha)$ satisfies the hypothesis above. Finally, fixing a torus \mathcal{T} does not affect generality, since all constant cocycles live in the fixed torus, modulo conjugation by a constant ([Die75], [DK00]). Since, by irrationality of α , the exponentials of such s are dense in \mathcal{T} , we have

$$\mathcal{AK}^\infty = cl^\infty\{\cup_{\alpha \in RDC}\{(\alpha, B(\cdot + \alpha).A.B^*(\cdot), A \in G, B(\cdot) \in C^\infty(\mathbb{T}, G))\}\}$$

Applying theorem 3.1 for each fixed $\alpha \in RDC$ gives that

$$\mathcal{AK}^\infty = cl^\infty\{\cup_{\alpha \in RDC} SW_\alpha^{\infty,1}(\mathbb{T}, G)\}$$

This last space by our first remark is $SW^{\infty,1}(\mathbb{T}, G)$. □

This corollary shows that the local picture as obtained in thm 3.3 is actually the corresponding part of the global picture, where uniquely ergodic, and therefore not reducible, cocycles form a generic subset in the space of cocycles. We remind, nonetheless, that we have allowed the frequency to vary (compare with thm 3.3, which holds for a fixed frequency, and in fact for an arbitrary number of frequencies). The analogy is, consequently, quite loose.

6 Almost reducibility and UE

In this section we present the basic points of the proof of the thm 3.2, which is the basis for proving thm 3.3. In fact, we will not (re-)prove this second theorem, since our goal is to prove a stronger one².

²Though in the C^∞ category, and not in analytic.

The rest of this section is devoted to proving thm 3.2 and analyzing the conjugations produced by the K.A.M. scheme used in its proof. From now on, $G = SU(2)$ and $d = 1$, even though the local almost reducibility theorem (and therefore also its direct consequences not requiring renormalization of the dynamics in their proof) is true for an arbitrary number of frequencies $d \in \mathbb{N}^*$.

6.1 Local conjugation

Let $(\alpha, Ae^{F(\cdot)}) = (\alpha, A_1 e^{F_1(\cdot)}) \in SW^\infty(\mathbb{T}, G)$ be a cocycle over a Diophantine rotation satisfying some smallness conditions to be made more precise later on. Without any loss of generality, we can also suppose that $A = \{e^{2i\pi a}, 0\}$ is diagonal. The goal is to conjugate the cocycle ever closer to constant cocycles by means of an iterative scheme. This is obtained by iterating the following lemma, for the detailed proof of which we refer to [Kri99], [Eli02] or [Kar13]. The rest of this paragraph is devoted to a summary of the proof, for the sake of completeness. The following lemma is the cornerstone of the procedure, since it represents one step of the scheme.

Lemma 6.1. *Let $\alpha \in DC(\gamma, \tau)$ and $K \geq C\gamma N^{\tau+1/2}$. Let, also, $(\alpha, Ae^{F_1(\cdot)}) \in SW^\infty(\mathbb{T}, G)$ with*

$$c_1 K N^{s_0} \varepsilon_{1,0} < 1$$

where c_0, s_0 depend on γ, τ (and d), and $\varepsilon_{1,s} = \|F_1\|_s$. Then, there exists a conjugation $G(\cdot) \in C^\infty(\mathbb{T}, G)$ such that

$$G(\cdot + \alpha).A_1.e^{F_1(\cdot)}.G^*(\cdot) = A_2.e^{F_2(\cdot)}$$

and such that the mappings $G(\cdot)$ and $F_2(\cdot)$ satisfy the following estimates

$$\begin{aligned} \|G(\cdot)\|_s &\leq c_{1,s}(N^s + KN^{s+1/2}\varepsilon_{1,0}) \\ \varepsilon_{2,s} &\leq c_{2,s}K^2N^{2\tau+1}(N^s\varepsilon_{1,0} + \varepsilon_{1,s})\varepsilon_{1,0} + C_{s,s'}K^2N^{s-s'+2\tau+1}\varepsilon_{1,s'} \end{aligned}$$

where $s' \geq s$.

If we suppose that $Y(\cdot) : \mathbb{T} \rightarrow g$ can conjugate $(\alpha, A_1 e^{F_1(\cdot)})$ to $(\alpha, A_2 e^{F_2(\cdot)})$, with $\|F_2(\cdot)\| \ll \|F_1(\cdot)\|$, then it must satisfy the functional equation

$$\begin{aligned} e^{Y(\cdot+\alpha)}A_1e^{F_1(\cdot)}e^{-Y(\cdot)} &= A_2e^{F_2(\cdot)} \Leftrightarrow \\ A_1^*e^{Y(\cdot+\alpha)}A_1e^{F_1(\cdot)}e^{-Y(\cdot)} &= A_1^*A_2e^{F_2(\cdot)} \end{aligned}$$

Linearization of this equation under the assumption that all C^0 norms are smaller than 1 gives

$$Ad(A_1^*)Y(\cdot + \alpha) + F_1(\cdot) - Y(\cdot) = \exp^{-1}(A_1^*A_2)$$

which we will write in the eigenspaces of $Ad(A_1^*)$, separating the diagonal from the non-diagonal part.

The equation for the diagonal coordinate reads

$$Y_t(\cdot + \alpha) - Y_t(\cdot) + F_{1,t}(\cdot) = 0$$

This equation can be solved by considering Fourier coefficients. For reasons well known in K.A.M. theory, we have to truncate at an order N to be determined by the parameters of the problem and obtain a solution to the equation

$$Y_t(\cdot + \alpha) - Y_t(\cdot) + \dot{T}_N F_{1,t}(\cdot) = 0$$

satisfying the estimate $\|Y_t(\cdot)\|_s \leq \gamma C_s N^{s+\tau+1/2} \varepsilon_{1,0}$. The rest satisfies the estimate of eq. 3. The mean value $\hat{F}_{1,t}(0)$ is an obstruction and will be integrated in $\exp^{-1}(A_1^* A_2)$.

As for the equation concerning the non-diagonal part, it reads

$$e^{-4i\pi a} Y_z(\cdot + \alpha) - Y_z(\cdot) + F_{1,z}(\cdot) = 0 \quad (6)$$

or, in the frequency domain,

$$(e^{2i\pi(k\alpha-2a)} - 1) \hat{Y}_z(k) = -\hat{F}_{1,z}(k), \quad k \in \mathbb{Z} \quad (7)$$

Therefore, the Fourier coefficient $\hat{F}_{1,z}(k_r)$ cannot be eliminated with good estimates if

$$|k_r \alpha - 2a|_{\mathbb{Z}} < K^{-1}$$

for some $K > 0$ big enough. If $K = N^\nu$, with $\nu > \tau$, then we know by [Eli02] (see also lem. 7.3) that such a k_r (called a *resonant mode*), if it exists and satisfies $0 \leq k_r \leq N$, is unique in $\{k \in \mathbb{Z}, |k - k_r| \leq 2N\}$. Therefore, if we call $T_{2N}^{k_r}$ the truncation operator projecting on the frequencies $0 < |k - k_r| \leq 2N$ if k_r exists, the equation

$$e^{-4i\pi a} Y_z(\cdot + \alpha) - Y_z(\cdot) = -T_{2N}^{k_r} F_{1,z}(\cdot)$$

can be solved and the solution satisfies $\|Y_z(\cdot)\|_s \leq C_s N^{s+\nu+1/2} \varepsilon_{1,0}$. We will define the rest operator in the obvious way. If k_r does not exist, we project on $|k| \leq 2N$, but this case more straightforward and left to the reader.

In total, the equation that can be solved with good estimates is

$$Ad(A_1^*)Y(\cdot + \alpha) - Y(\cdot) + F_1(\cdot) = \{\hat{F}_{1,t}(0), \hat{F}_{1,z}(k_r) e^{2i\pi k_r \cdot}\} + \{R_N F_{1,t}(\cdot), R_{2N}^{k_r} F_{1,z}(\cdot)\}$$

with $\|Y(\cdot)\|_s \leq C_s N^{s+\nu+1/2} \varepsilon_{1,0}$. Under the smallness assumptions of the hypothesis, the linearization error is small and the conjugation thus constructed satisfies

$$e^{Y(\cdot + \alpha)} A_1 e^{F_1(\cdot)} e^{-Y(\cdot)} = \{e^{2i\pi(a + \hat{F}_{1,t}(0))}, 0\}_G . e^{\{0, \hat{F}_{1,z}(k_r) e^{2i\pi k_r \cdot}\}_g} e^{F_2'(\cdot)}$$

with $F_2'(\cdot)$ a "quadratic" term. We remark that, a priori, the obstruction $\{0, \hat{F}_{1,z}(k_r) e^{2i\pi k_r \cdot}\}$ is of the order of the initial perturbation and therefore what we called $\exp^{-1}(A_1^* A_2)$ is not constant in the presence of resonant modes.

If k_r exists and is non-zero, iteration of the lemma as it has been proved so far is impossible. On the other hand, the conjugation $B(\cdot) = \{e^{-2i\pi k_r \cdot / 2}, 0\}$ is such that, if we call $F'_1(\cdot) = \text{Ad}(B(\cdot)).F_1(\cdot) = \{F_{1,t}(\cdot), e^{-2i\pi k_r \cdot} F_{1,z}(\cdot)\}$, similarly for $Y(\cdot)$, and $A'_1 = B(\alpha)A_1 = \{e^{2i\pi(a-k_r\alpha/2)}\}$, they satisfy the equation

$$\begin{aligned} \text{Ad}((A')^*)Y'(\cdot + \alpha) - Y'(\cdot) + F'_1(\cdot) &= \{\hat{F}_t(0), \hat{F}_z(k_r)\} + \{R_N F_t(\cdot), e^{-2i\pi k_r \cdot} R_{2N}^{k_r} F_z(\cdot)\} \\ &= \{\hat{F}'_{1,t}(0), \hat{F}'_{1,z}(0)\} + \{R_N F'_{1,t}(\cdot), \tilde{R}_{2N}^{k_r} F_{1,z}(\cdot)\} \end{aligned}$$

where $\tilde{R}_{2N}^{k_r}$ is a dis-centered rest operator, whose spectral support is outside $[-N, N] \cap \mathbb{Z}$, and can therefore be estimated like a classical rest operator R_N . The equation for primed variables can be obtained from eq. 6 by applying $\text{Ad}(B(\cdot))$ and using that $B(\cdot)$ is a morphism and commutes with A_1 . The passage from one equation to the other is equivalent to the fact that

$$\begin{aligned} \text{Conj}_{B(\cdot)}(\alpha, A_1 \cdot \exp(\{\hat{F}_t(0), \hat{F}_z(k_r)e^{2i\pi k_r \cdot}\})) &= (\alpha, A'_1 \cdot \exp(\{\hat{F}_t(0), \hat{F}_z(k_r)\})) \\ &= (\alpha, A_2) \end{aligned}$$

that is, $B(\cdot)$ reduces the initial constant perturbed by the obstructions to a cocycle close to $(\alpha, \pm Id)$. There is a slight complication, as $B(\cdot)$ may be 2-periodic. If it is so, we can conjugate a second time with a minimal geodesic $C(\cdot) : 2\mathbb{T} \rightarrow G$ such that $C(1) = -Id$ and commuting with A' . The cocycle that we obtain in this way is close to $(\alpha, \{e^{i\pi\alpha}, 0\}_G)$ and the conjugation is 1-periodic. We will omit this conjugation from now on, since it does not affect the estimates.

6.2 The K.A.M. scheme

Lemma 6.1 can serve as the step of a K.A.M. scheme, with the following standard choice of parameters: $N_{n+1} = N_n^{1+\sigma} = N^{(1+\sigma)^{n-1}}$, where $N = N_1$ is big enough and $0 < \sigma < 1$, and $K_n = N_n^\nu$, for some $\nu > \tau$. If we suppose that $(\alpha, A_n e^{F_n(\cdot)})$ satisfies the hypotheses of lemma 6.1 for the corresponding parameters, then we obtain a mapping $G_n(\cdot) = B_n(\cdot)e^{Y_n(\cdot)}$ that conjugates it to $(\alpha, A_{n+1} e^{F_{n+1}(\cdot)})$, and we use the notation $\varepsilon_{n,s} = \|F_n\|_s$.

If we suppose that the initial perturbation small in small norm: $\varepsilon_{1,0} < \epsilon < 1$, and not big in some bigger norm: $\varepsilon_{1,s_0} < 1$, where ϵ and s_0 depend on the choice of parameters, then we can prove (see [Kar13] and, through it, [FK09]), that the scheme can be iterated, and moreover

$$\begin{aligned} \varepsilon_{n,s} &= O(N_n^{-\infty}) \text{ for every fixed } s \text{ and} \\ \|G_n\|_s &= O(N_n^{s+\lambda}) \text{ for every } s \text{ and some fixed } \lambda > 0 \end{aligned}$$

We say that the norms of perturbations decay exponentially, while conjugations grow polynomially.

6.3 Some remarks and consequences

The product of conjugations $H_n = G_n \cdots G_1$, which by construction satisfies $\text{Conj}_{H_n}(\alpha, A_1 e^{F_1}) = (\alpha, A_{n+1} e^{F_{n+1}})$, is not expected to converge. In fact, it

converges iff $B_n(\cdot) \equiv Id$, except for a finite number of steps. We refer to [Kar14] for the proof of the convergence of H_n if the cocycle is L^2 -conjugate to a constant one, whose eigenvalues satisfy a Diophantine condition with respect to α ; and of the "generic" divergence of H_n if the eigenvalues are Liouville with respect to α . Therefore, reducibility cannot be ruled out by the divergence of H_n , at least without the further analysis of the scheme carried out in the following section (sec. 6.4). Anyhow, we can obtain a "K.A.M. normal form" for cocycles close to constants

Lemma 6.2. *Let the hypotheses of theorem 3.2 hold. Then, there exists $D(\cdot) \in C^\infty(\mathbb{T}, G)$ such that, if we call $Conj_D(\alpha, Ae^F) = (\alpha, A'e^{F'})$, then the K.A.M. scheme applied to $(\alpha, A'e^{F'})$ for the same choice of parameters consists only in the reduction of resonant modes: The resulting conjugation $H_{n_i}(\cdot)$ has the form $\prod_i^1 B_{n_j}(\cdot)$, where $\{n_i\}$ are the steps in which reduction of a resonant mode took place.*

In fact, we implicitly suppose an intelligent scheme, where, if we separate the perturbation $e^{F_n(\cdot)}$ in $e^{F_{n,res}(\cdot)} \cdot e^{F_{n,nr}(\cdot)}$, the resonant and non-resonant part, and the non-resonant part satisfies the inductive smallness conditions of step $n+1$, the scheme produces only the reduction of the resonant part. Of course, nothing changes as to the significant part of the scheme.

Proof. The product of conjugations produced by the scheme at the n -th step is written in the form $H_n(\cdot) = B_n(\cdot)e^{Y_n(\cdot)} \dots B_1(\cdot)e^{Y_1(\cdot)}$, where the $B_j(\cdot)$ reduce the resonant modes. We can rewrite the product in the form

$$B_n(\cdot) \dots B_1(\cdot) \cdot e^{\tilde{Y}_n(\cdot)} \dots e^{\tilde{Y}_2(\cdot)} e^{Y_1(\cdot)}$$

where $\tilde{Y}_j(\cdot) = \prod_{i=j}^1 Ad(B_i^*(\cdot)) \cdot Y_j(\cdot)$. Since the $Y_j(\cdot)$ converge exponentially fast to 0 (they are conjugations comparable with F_j with a fixed loss of derivatives) in C^∞ , and since the algebraic conjugation deteriorates the C^s norms by a factor of the order of N_{n-1}^{s+1} , $\prod_{i=1}^n \exp(\tilde{Y}_i(\cdot))$ always converges, even if the $H_n(\cdot)$ do not. The rest of the properties follow by construction. \square

Notation 1. *For a cocycle in normal form, we relabel the indexes as $(\alpha, A_{n_i} e^{F_{n_i}}) = (\alpha, A_i e^{F_i})$.*

The lemma therefore asserts that any cocycle on which the K.A.M. scheme of the previous paragraph can be applied can be conjugated to a cocycle for which the following holds. At every step of the scheme, the path $A_i e^{F_i}$ is such that the argument of the eigenvalues, a_i , is close to some $k_i \alpha / 2$, and therefore not 0, so that a direction in g is defined by it. To the first order, the image of the path $A_i e^{F_i}$ is a circle centered on A_i and contained in the plane orthogonal to the direction of a_i , and it is traversed k_i times when x goes once around \mathbb{T} . The significant parameters in this expression are: k_i , $\epsilon_i = a_i - k_i \alpha / 2 \mod \mathbb{Z}$ and $|\hat{F}_i(k_i)|$. The argument of the complex number $\hat{F}_i(k_i)$ is not important, since, if $D \in G$ commutes with A_i , $Conj_D(\alpha, A_i e^{F_i}) = (\alpha, A_i e^{Ad(D) \cdot F_i})$ and the perturbation term has a Fourier coefficient with the same magnitude, but a

different argument. The change of coordinates $(x, S) \mapsto (x, B_i(x).S)$ sends this circle to the constant,

$$\begin{aligned} A_{i+1} &= \{e^{2i\pi\epsilon_i}, 0\}_G . e^{\{0, \hat{F}_i(k_i)\}_g} \\ &= e^{\{2\pi\epsilon_i, \hat{F}_i(k_i)\}_g + O(\epsilon_i \cdot |\hat{F}_i(k_i)|)} \end{aligned}$$

so that $\text{dist}_G(D, \mathcal{N}_i) \simeq 1 - \cos(\frac{1}{2} \arctan |\frac{\epsilon_i}{\hat{F}_i(k_i)}|)$. The hypothesis that the cocycle is in normal form implies that a_{i+1} is $K_{n_{i+1}}^{-1}$ -close to some resonant sphere of radius $|k_{i+1}\alpha/2|_{\frac{1}{2}\mathbb{Z}}$.

This K.A.M. normal form allows us to discuss the genericity of UE for C^∞ -smooth cocycles. H. Eliasson's condition as he stated it becomes, in our context,

$$\liminf \frac{|\langle LB_i, LB_{i+1} \rangle|}{|LB_i| \cdot |LB_{i+1}|} < 1 \quad (8)$$

i.e. the reductions of resonant modes of consecutive steps of the algorithm form an angle bigger than some $\delta > 0$, fixed throughout the scheme, for an infinite number of steps. Since LB_i points in the direction of A_i , an equivalent statement is that $\limsup \frac{d(A_{i+1}, \mathcal{N}_i)}{d(A_{i+1}, Id)} > \delta$, possibly for a different $\delta > 0$, but of comparable size. Here, $d(\cdot, \mathcal{N}_i)$ is the Riemannian distance from the normalizer of the torus of matrices simultaneously diagonalizable with A_i , and d is the Riemannian distance on G . This last expression says exactly that if $D_i \in G$ diagonalizes A_{i+1} in a basis where A_i is diagonal, then D_i is δ -away from \mathcal{N}_i , for an infinite number of steps.

For a cocycle in K.A.M normal form, this quantity admits a straightforward geometric interpretation. At the i -th step of the algorithm, we can write the cocycle in the form

$$(\alpha, A_i e^{F_i(\cdot)}) = (\alpha, \{e^{2i\pi a_i}, 0\}_G . e^{\{0, \hat{F}_i(k_i) e^{2i\pi k_i \cdot}\}_g} . e^{F'_i(\cdot)})$$

where $|k_i\alpha - 2a_i|_{\mathbb{Z}} < K_{n_i}^{-1}$, so that k_i is the resonant mode and we write $a_i = k_i\alpha/2 + \epsilon_i + l_i/2$, $l_i \in \mathbb{Z}$,³ and $\|F'_i\|_0$ is much smaller (in fact exponentially) than $|\hat{F}_i(k_i)|$, which is supposed strictly positive.

The condition for UE takes, therefore, the equivalent form

$$\liminf \left| \frac{\epsilon_i}{\hat{F}_i(k_i)} \right| < \infty$$

We remark already that this condition implies that the corresponding small divisor of eq. 7 is comparable with $|\hat{F}_i(k_i)|$. Since the resonant modes $\hat{F}_i(k_i)$ are (essentially) the Fourier coefficients of a C^∞ mapping, $|\hat{F}_i(k_i)| = O(N_{n_i}^{-\infty})$, which implies that under the condition of eq. 8, $|\epsilon_i|_{\mathbb{Z}} = O(N_{n_i}^{-\infty})$ (at least along a subsequence of $\{n_i\}$), so that the estimation $|\epsilon_i|_{\mathbb{Z}} \leq K_{n_i}^{-1} = N_{n_i}^{-\nu}$ is

³In fact, $l_i = l$ is constant mod 2 throughout the scheme, and we suppose that it is even for simplicity. If it is odd, the limit cocycle is $(\alpha, -Id)$, instead of (α, Id) . This indeterminacy does not occur when $G = SO(3)$.

very optimistic in this case. In other words, a cocycle is UE iff the rotation around which we linearize at an infinite number of steps of the K.A.M. scheme is (exactly resonant or) Liouville with respect to α . We think that the genericity of condition of eq. 8 is now clear.

The K.A.M. normal form of the cocycle also allows us to describe quite explicitly a UE cocycle. Let us fix $D \in G$ a matrix conjugating $\{1, 0\}_g$ to $\{0, 1\}_g$, and fix $\{q_{n_i}\}$ a sub-sequence of good denominators of α , with n_i increasing fast enough. We omit the notation of the sub-sequence. Then, the cocycle

$$(\alpha, \{e^{2i\pi q_1 \alpha}, 0\}_{G \cdot e^{\{0, \beta_2 e^{2i\pi 2q_1 \cdot}\}_g}})$$

is resonant for the first step of K.A.M. The reduction of the resonant mode by $B_1 = \{e^{-2i\pi q_1 \cdot}, 0\}$ conjugates it to the resonant constant

$$(\alpha, D\{e^{2i\pi q_2 \alpha}, 0\}D^*)$$

Therefore, if we perturb the initial cocycle by

$$\exp(Ad(D.B_1(\cdot)D^*). \{0, \beta_3 e^{2i\pi 2q_2 \cdot}\}_g)$$

it will be resonant in the next step. Then, iterating the construction we can obtain a cocycle satisfying the smallness conditions of the K.A.M. scheme at all steps, as well as Eliasson's condition for UE. Such cocycles are actually cohomologically stable, as will be proved in section 9.

In order to obtain a more general UE cocycle, one can replace D by a sequence $\{D_i\}$ of matrices such that D_{i+1} is away from all matrices simultaneously diagonalizable with D_i , and q_i by a fast increasing sequence $\{k_i\}$ such that $|k_i \alpha|_{\mathbb{Z}}$ decreases exponentially fast. Such cocycles, as well as typical cocycles close to them, are UE.

Finally, we discuss the rareness of reductions of resonances, following [Eli02]. Since A_{i+1} is $K_{n_i}^{-1} = N_{n_i}^{-\nu}$ -close to the identity, the resonant mode k_{i+1} must satisfy

$$\frac{\gamma^{-1}}{|k_{i+1}|^\tau} \leq N_{n_i}^{-\nu}$$

so that $|k_{i+1}| \geq \gamma^{-1/\tau} N_{n_i}^{\nu/\tau}$. Since $|k_{i+1}| \leq N_{n_{i+1}} = N_{n_i}^{(1+\sigma)n_{i+1}-n_i-1}$, we obtain that

$$n_{i+1} - n_i \gtrsim \frac{\log \nu - \log \tau}{\log(1 + \sigma)}$$

where we remind that σ is chosen positive and small. The estimate is of course worse than the exponential gaps for analytic cocycles. The gaps in C^∞ are not expected to be exponential. They are, however, under condition 8.

6.4 Local density of H^s -reducible cocycles and optimality of the K.A.M. scheme

We consider a cocycle $(\alpha, A_1 e^{F_1(\cdot)}) \in \mathcal{W}$ given in K.A.M. normal form and fix $0 \leq \sigma < \infty$. The case where $\sigma = \infty$ (i.e. the case where the scheme does not

produce a converging product of conjugations even if there exists one) we be deduced from this one. If the normal form is a finite product of resonant modes (in which case the cocycle is reducible), the cocycle can be perturbed so that the product becomes infinite. We can now proceed to the proof of theorem 1.1.

We first state the condition on the angles between successive constants and prove its density. It is obtained by imposing a rate of degeneracy of condition 8. To this end, we fix i_0 big enough so that the the tail of the normal form is small in C^∞ . We also fix a sequence $e \in \ell^2$ such that $e_i \neq 0$, for all the steps of the scheme. If we perturb the constants $A_i, i \geq i_0$ to \tilde{A}_i so that the corresponding \tilde{e}_i satisfy

$$|\tilde{e}_i| = N_{n_i}^\sigma \frac{\|F_i\|_{L^2}}{|e_i|} \ll N_{n_i}^{-\nu}$$

which is equivalent to the fact that the series with general term

$$\{\delta_i\} = \left\{ \frac{\|F_i\|_{L^2}}{|\tilde{e}_i|} \right\}_{i \geq i_0} \in \tilde{h}^\sigma \quad (9)$$

(in fact, $\|\delta\|_{h^\sigma} = \|e\|_{\ell^2}$). The perturbation can be made arbitrarily small in C^∞ by fixing the sequences e and $\{\|F_i\|_{L^2}\}$ and choosing i_0 big enough.

We now construct a conjugation of regularity H^σ reducing the cocycle.

Lemma 6.3. *The condition of eq. 9 is satisfied iff there exists a conjugation $H(\cdot)$ of regularity H^σ and a constant $A' \in SU(2)$ such that*

$$Conj_{H(\cdot)}(\alpha, A_1 e^{F_1(\cdot)}) = (\alpha, A')$$

Proof. Let $i > i_0$, and recall that

$$Conj_{H_{i-1}}(\alpha, A_1 e^{F_1}) = (\alpha, A_i e^{F_i}), \text{ and } Conj_{B_i}(\alpha, A_i e^{F_i}) = (\alpha, A_{i+1} e^{F_{i+1}})$$

The assumption on δ_i implies that, if D_i diagonalizes A_{i+1} in the basis where A_i is diagonal, then D_i is not away from the identity in $SU(2)/\mathcal{T}_i$. Then, the conjugation $G'_i = B_i^*(\cdot) D_i B_i(\cdot)$ is $N_i^s |\delta_i|$ -away from the Id in H^s . Moreover, as shows a direct calculation, up to the first order,

$$Conj_{G'_i}(\alpha, A_i e^{F_i}) = (\alpha, \{e^{2i\pi(k_{i+1}\alpha + \epsilon_{i+1})}, 0\} \cdot \exp\{0, \hat{F}_{i+1}(k_{i+1}) e^{2i\pi(k_{i+1} + k_i)\cdot}\})$$

We can therefore continue the reduction by defining

$$\begin{aligned} \tilde{B}_i(\cdot) &= \exp\{2i\pi \sum_{j=i_0}^i k_j \cdot, 0\} \\ D_i &\in SU(2)/\mathcal{T}_{i_0}, D_i A_i D_i^* \in \mathcal{T}_{i_0} \\ G'_i(\cdot) &= \tilde{B}_i^*(\cdot) D_i \tilde{B}_i(\cdot) \\ H'_i(\cdot) &= \prod_{j=i_0}^i G'_j(\cdot) \end{aligned}$$

where \mathcal{T}_{i_0} is the torus of matrices simultaneously diagonalizable with A_{i_0} . Then, $H'_i(\cdot) \rightarrow H(\cdot)$ in H^σ and not in higher regularity.

The proof of the converse follows the lines of the direct implication and is left to the reader. \square

A simple perturbation argument allows us to obtain the following corollary (cf [Cha12]).

Corollary 6.4. *Let $\alpha \in DC(\gamma, \tau)$, and $\sigma \geq s_0$, where s_0 is given by the smallness condition of the local almost reducibility theorem. Consider the set of C^σ -smooth cocycles over the diophantine rotation $\alpha \in \mathbb{T}^d$, denoted by $SW_\alpha^\sigma(\mathbb{T}^d, SU(2))$ and \mathcal{W} , the neighborhood of constants as in theorem 1.1. Then, $C^{\sigma-d/2}$ -reducible cocycles are $C^{\sigma-d/2}$ dense in \mathcal{W} .*

Proof. We perturb the given cocycle in C^σ so that it becomes C^∞ , and we apply theorem 1.1. The Sobolev injection theorem accounts for the loss of derivatives. \square

7 DUE in $SW_\alpha^\infty(\mathbb{T}, SU(2))$

We now prove thm 1.3. The strategy of the proof is to use the equivalent definition of DUE, namely that "the closure of coboundaries in C^∞ equals the space of 0-average functions", and we will prove that under a condition of the flavour of the one of eq. 8 this holds. The setting is therefore that of theorem 3.3, which we place in $\mathbb{T} \times SU(2)$ for convenience in calculations, and in the discrete-time setting, in accordance with our proof of thm 3.2.

We think that, in view of the normal form obtained in lemma 6.2, the proof of local genericity of DUE also clarifies to a certain extent the proof of local genericity of unique ergodicity. Roughly, the proof consists in using the K.A.M. scheme of the local almost reducibility theorem in order to solve

$$\psi \circ \Phi - \psi = \phi - \int \phi$$

up to an arbitrarily small error in $C^\infty(\mathbb{T} \times SU(2))$, where $\phi \in C^\infty$ is a given function (from now on of 0 mean value) and $\Phi = \Phi_1 = (\alpha, A_1 e^{F_1(\cdot)}) \in SW_\alpha^\infty$ is a given cocycle close to constants satisfying a generic condition. The equation can be solved by linearizing with respect to the dynamics (which introduces an error term), and solving for a finite truncation (both in frequencies and the harmonics). The linearized equation has a large number of obstructions, the mappings in the kernels of distributions preserved by a constant (moreover, resonant) cocycle. We solve, therefore,

$$\psi_1 \circ \Phi_1 - \psi_1 - \phi = R_1 \phi + Ob_1 \phi + E_1 = \phi_2$$

where R_1 is a rest operator, Ob_1 the projection on the obstructions, and $E_1 = O_{C^s}(\|\partial \psi_1\|_s \|F_1\|_s)$, for every s . When we make one step of the K.A.M. scheme

and introduce the resulting change of coordinates we obtain the cocycle $(\alpha, A_2 e^{F_2(\cdot)})$ which is a perturbation of a different constant, and the order of the perturbation is much smaller. We now obtain the equation

$$\psi'_2 \circ \Phi_2 - \psi'_2 = \phi'_2 = R'_1 \phi + Ob'_1 \phi + E'_1$$

where the prime indicates that quantities are expressed in the new coordinates. Since F_2 is quadratic, the error term will be smaller, allowing more frequencies and harmonics in the truncation (and a larger space of obstructions). As for $Ob'_1 \phi$, they are constant in x . If, now, A_2 does not commute with A_1 , lemma 2.1 says that only a fraction (in the L^2 norm) of these obstructions persists as an obstruction to the new equation. We thus solve

$$\psi'_2 \circ \Phi_2 - \psi'_2 = Ob'_2 \phi'_2 + R'_2 \phi'_2 + E'_2$$

When we inverse the change of coordinates, we find that $Ob_2 Ob'_1 \phi$, in the standard coordinates, has its spectral support in the same ball as Ob_1 , but a smaller L^2 , and therefore H^s , norm. When we change coordinates following the 3rd step of K.A.M., $(Ob_2 Ob'_1)^{(2)} \phi$ will still not depend on x in the new coordinates, and $Ob_3 (Ob_2 Ob'_1)^{(2)} \phi$ will have a smaller L^2 norm for the same reason. Inversion of the change of coordinates, however, will create new frequencies. We need, therefore, to impose a condition guaranteeing that the spaces of obstructions of successive constants are almost orthogonal infinitely often for every fixed m . This is the refinement of condition 8 needed in order to assure DUE.

We thus obtain that the closure of coboundaries coincides with 0 average functions, which is equivalent to saying that the only distribution preserved by the dynamics is the product of the Haar measures.

Another interpretation of the proof could be the following. The K.A.M. scheme, when the product of conjugacies diverges, can be seen as a formal solution to the equation (with respect to $B_\infty(\cdot)$)

$$Conj_{B_\infty(\cdot)}(\alpha, A e^{F(\cdot)}) = (\alpha, Id)$$

This is so, because the sequence of constants constructed by the scheme then converges to the Id and the perturbation converges to 0 in C^∞ . The condition assures that, when we invert the scheme in order to obtain the original cocycle from its formal limit, the Fourier coefficients of all the invariant distributions are driven to infinity by the conjugations reducing the resonant modes.

Cocycles preserving only finite distributions in the border between DUE and UE do not exist, since if we let the condition assuring DUE degenerate, distributions of arbitrarily high order are created, and no linear dependence between their Fourier coefficients can be forced. This phenomenon has, however, been observed as in a space of abelian extensions of Liouvillean rotations by A. Avila and A. Kocsard, in a forthcoming article.

7.1 The basic lemmas

We suppose that the cocycle is given in normal form. Since the K.A.M. scheme produces only conjugations far from the identity, we study the behavior of har-

monic functions on $\mathbb{T} \times G$ under such conjugations. The proof is just a calculation.

Lemma 7.1. *If $B(\cdot) : \mathbb{T} \rightarrow SU(2)$ is of the form*

$$\begin{pmatrix} e^{2i\pi k_0 x} & 0 \\ 0 & e^{-2i\pi k_0 x} \end{pmatrix}$$

then the harmonic $e^{2i\pi kx} \pi_m^{j,p}(z, \bar{z}, w, \bar{w})$ is mapped to $e^{2i\pi(k+(m-2p)k_0)x} \pi_m^{j,p}(z, \bar{z}, w, \bar{w})$.

As a consequence, we obtain the following crucial estimate.

Lemma 7.2. *Let $\phi(\cdot) \in C^\infty(\mathbb{T} \times SU(2))$, $B(\cdot)$ as in the previous lemma, and call*

$$\tilde{\phi}(x, \{z, w\}) = \phi(x, B^{-1}(x). \{z, w\})$$

Then

$$\|\tilde{\phi}\|_s \leq C_s(k_0)^{s+2} \|\phi\|_0$$

Proof. The proof follows the lines of similar estimations, using the expansion $\phi(x, \{z, w\}) = \sum_{k \in \mathbb{Z}, m \in \mathbb{N}, 0 \leq j, p \leq m} \phi_{k,j,p}^m e^{2i\pi kx} \pi_m^{j,p}$

$$\begin{aligned} \|\tilde{\phi}\|_s^2 &\leq C_s \sum (1 + k^2 + m^2)^{s+2} |\tilde{\phi}_{k,j,p}^m|^2 \\ &= C_s \sum (1 + (k + (m-2p)k_0)^2 + m^2)^{s+2} |\phi_{k,j,p}^m|^2 \\ &\leq C_s k_0^{2(s+2)} \sum (1 + k^2 + m^2)^{s+2} |\phi_{k,j,p}^m|^2 \\ &\leq C_s k_0^{2(s+2)} \|\phi\|_0^2 \end{aligned}$$

where as usually the constant may change from one line to another. \square

Moreover, after applying such a conjugation, the C^s norms of harmonics deteriorate by factors like k_0^s .

Let us now return to the cohomological equation that we seek to solve, supposing that the cocycle is directly given in normal form. We remind that $\phi \in C_0^\infty$ (we suppose that $\int \phi = 0$) can be written as

$$\sum_{\substack{k \in \mathbb{Z}, m \geq 0, 0 \leq j, p \leq m \\ \{k, m\} \neq \{0, 0\}}} \phi_{k,j,p}^m e^{2i\pi kx} \pi_m^{j,p}$$

where $x \in \mathbb{T}$ and $\{z, w\} \in \mathbb{S}^3 \approx G$.

The cohomological equation over a constant cocycle (α, A) reads

$$\psi(x + \alpha, A^* \cdot \{z, w\}) - \psi(x, \{z, w\}) = \phi(x, \{z, w\})$$

In terms of coefficients, and for $A = \{e^{2i\pi a}, 0\}$, we have

$$(e^{2i\pi(k\alpha + (m-2p)a)} - 1) \psi_{k,j,p}^m = \phi_{k,j,p}^m$$

Thus, the resonant modes (the obstructions of the equation), are given by the relation $k\alpha + (m - 2p)a \in \mathbb{Z}$. In view of the K.A.M. normal form, the resonance condition is implied by

$$a = \frac{k_0}{2}\alpha \in \frac{1}{2}\alpha\mathbb{Z} \text{ and } k + \frac{1}{2}(m - 2p)k_0 = 0$$

This relation is, of course, valid in a coordinate system $\{z, w\} \in \mathbb{S}^3$ where A is diagonal. For a fixed vector $e^{2i\pi kx} z^p w^{m-p}$ (where m, p are fixed and $k \in \mathbb{Z}$) there is at most one solution to the equation [10](#).

The cohomological equation for the perturbed cocycle reads

$$\psi(x + \alpha, e^{-F(\cdot)} A^* \cdot \{z, w\}) - \psi(x, \{z, w\}) = \phi(x, \{z, w\})$$

which we rewrite as

$$\psi(x + \alpha, A^* \cdot \{z, w\}) - \psi(x, \{z, w\}) = \phi(x, \{z, w\}) + O(\|\partial\psi\|_s \|F\|_s)$$

where the last term is a linearization error term. This fact obviously poses a constraint on the number of modes that can be reduced, relatively to the size of the perturbation. The resonance relation holds approximately in this setting, since the cocycle is a perturbation of a resonant constant, $(\alpha, Ae^{F(\cdot)})$ where $a = \frac{k_0}{2}\alpha + \epsilon$, with $|\epsilon| < K^{-1}$. We have $|\hat{F}(k_0)| \ll K^{-1}$, and our condition (condition [1](#)) will imply that $|\epsilon| \lesssim |\hat{F}(k_0)|$ infinitely often in the K.A.M. scheme. Therefore, the resonant modes, for which the small denominator is $< K^{-1}$, are

$$k + (m - 2l)\frac{k_0}{2} = 0, 0 < m < N \quad (10)$$

where N is the corresponding truncation order in the K.A.M. scheme, and $K = N^\nu$. We recall the following lemma from [\[Eli02\]](#).

Lemma 7.3. *Let $\alpha \in DC(\gamma, \tau)$, $\beta \in \mathbb{T}$, $N \in \mathbb{N}^*$ big enough, and $K = N^\nu > 0$, with $\nu > \tau$, so that $K \geq 2^{\tau+1}\gamma N^\tau$. Then there exists at most one $k = k_0$, $0 \leq |k_0| \leq N$, such that*

$$|\beta - k\alpha|_{\mathbb{Z}} < K^{-1}, \quad 0 < |k| \leq N$$

Moreover, if such a k_0 exists and $|\beta - k_0\alpha|_{\mathbb{Z}} = \epsilon$

$$|\beta - k\alpha|_{\mathbb{Z}} \geq K^{-1}, \quad 0 < |k - k_0| \leq N'$$

$$\text{for } N' = \left(\frac{\gamma}{1+K\epsilon}\right)^{1/\tau} N^{\nu/\tau} \geq \left(\frac{\gamma}{2}\right)^{1/\tau} N^{\nu/\tau}.$$

Therefore, if the resonance relation is satisfied with an ϵ of room, as in the lemma, and if we call $T_{N'}^{k_0}$ the operator acting on $C^\infty(\mathbb{T} \times G)$ and projecting on the modes (k, j, p, m) such that $0 < |k - (m - 2p)\frac{k_0}{2}| \leq N'$,

$$T_{N'}^{k_0} \phi(x, \{z, w\}) = \sum_{0 < |k + (m - 2p)\frac{k_0}{2}| \leq N'} \phi_{k,j,p}^m e^{2i\pi kx} \pi_m^{j,p}$$

the denominators in the image of $T_N^{k_0}$ are $\geq K^{-1}$. On the other hand, if $Ob_N^{k_0}$ is the projection on the resonant modes,

$$Ob_N^{k_0}\phi(x, \{z, w\}) = \sum_{k+(m-2p)\frac{k_0}{2}=0, |k|\leq N'} \phi_{k,j,p}^m e^{2i\pi kx} \pi_m^{j,p}$$

the denominators in the image of $Ob_N^{k_0}$ are $< K^{-1}$, and the corresponding modes cannot be eliminated with good estimates. Therefore, the equation

$$\psi(x + \alpha, A^*.\{z, w\}) - \psi(x, \{z, w\}) = T_N^{k_0}\phi(x, \{z, w\})$$

admits a solution that has the same harmonics as $T_N^{k_0}\phi(x, \{z, w\})$ and whose coefficients satisfy $|\psi_{k,j,p}^m| \leq K|\phi_{k,j,p}^m|$. Therefore, this solution satisfies the estimate

$$\|\psi\|_s \leq KN^{s+3}\|\phi\|_0$$

Consequently, if the rest operator $R_N^{k_0}$ is defined so that

$$T_N^{k_0} + Ob_N^{k_0} + R_N^{k_0} = Id$$

we have obtained a solution to the equation

$$\psi(x + \alpha, A^*.\{z, w\}) - \psi(x, \{z, w\}) = Ob_N^{k_0}\phi(x, \{z, w\}) + R_N^{k_0}\phi(x, \{z, w\})$$

where of course, the obstruction is of the order of ϕ , and the rest is small:

$$\begin{aligned} \|Ob_N^{k_0}\phi\|_s &\leq C_s(N')^{s+2}\|\phi\|_0 \\ \|R_N^{k_0}\phi\|_s &\leq C_{s,s'}(N')^{s-s'+2}\|\phi\|_{s'} \end{aligned}$$

Finally, the error term coming from the linearization can be estimated by

$$\begin{aligned} \|\psi(x + \alpha, A^*.\{z, w\}) - \phi(x + \alpha, e^{-F(\cdot)}A^*.\{z, w\})\|_s &\leq C_s(N')^2\|\psi\|_{s+1}\|F\|_s \\ &\leq C_sK(N')^{s+4}\|\phi\|_0\|F\|_s \end{aligned}$$

Therefore, we obtain

Lemma 7.4. *If $\Phi = (\alpha, A^{F(\cdot)})$ is in normal form and $\phi \in C_0^\infty$, then the mapping ψ constructed above satisfies*

$$\psi \circ \Phi(x, \{z, w\}) - \psi(x, \{z, w\}) - \phi(x, \{z, w\}) = Ob_N^{k_0}\phi(x, \{z, w\}) + R_N^{k_0}\phi(x, \{z, w\}) + E(x, \{z, w\})$$

where

$$\begin{aligned} \|E\|_s &\leq C_sK(N')^{s+4}\|\phi\|_0\|F\|_s \\ \|R_N^{k_0}\phi\|_s &\leq C_{s,s'}(N')^{s-s'+3}\|\phi\|_{s'} \\ \|\psi\|_s &\leq KN^{s+3}\|\phi\|_0 \end{aligned}$$

7.2 Construction of the approximate solution

It is in this point that a condition concerning successive constants, like condition of eq. 8, becomes important. The reason is clearly the fact that given a size of a perturbation, we can solve the cohomological equation up to a certain error, and by excluding the resonant harmonics which we have called obstructions. The error converges to 0 faster than the rate of growth of the conjugations, so this non-resonant part is easy to cope with, thanks to the fast convergence of the K.A.M. scheme. In order to cope with the obstructions, we need to impose a condition assuring that obstructions do not accumulate. They accumulate, for example, if the cocycle is reducible, and the scheme will show an accumulation of obstructions if, for example, the cocycle is C^∞ reducible to a Liouvillean constant, in which case the scheme generically produces an infinite number of reductions of resonances, due to the arithmetics of the dynamics in the fibers. We will see that obstructions will not accumulate if the reduction of resonances visits the angles corresponding to the roots of Legendre polynomials (cf. lemma 2.2) infinitely often. This is stronger than imposing that that constants around which we linearize in successive steps of K.A.M. do not commute as in eq. 8.

In this section, we will settle in a first time the non-resonant modes, something that can be done without complications since the equation we are trying to solve is linear. Then, we will show that condition 1, which is a dynamical condition stated in terms of the K.A.M. scheme, prevents obstructions from accumulating, so that in total we obtain an approximate solution to the cohomological equation. We have already proved the following lemma.

Lemma 7.5. *Let $\Phi = (\alpha, Ae^{F(\cdot)})$ satisfy the resonance and smallness conditions of the K.A.M. scheme. Then, if ϕ has no obstructions, i.e. if $Ob\phi = 0$, the equation*

$$\psi \circ \Phi - \psi = T_N \phi$$

admits an approximate solution satisfying $\|\psi\| \leq C_s K N^{s+3} \|\phi\|_0$. The error term, caused by linearization in the dynamics, can be estimated by

$$C_s K N^{s+4} \|F\|_s \|\phi\|_0$$

The next lemma concerns the elimination of resonant modes. This requires iteration of the K.A.M. scheme.

Lemma 7.6. *Let $(\alpha, A_1 e^{F_1(\cdot)} e^{F'_1(\cdot)})$ be a cocycle and ϕ a function satisfying*

1. $A_1 = \{e^{2i\pi(k_1\alpha + \epsilon_1)/2}, 0\} \in G$ is a resonant constant, up to $|\epsilon_1| \leq N_1^{-\nu}$, and the resonant mode is k_1 , with $|k_1| \leq N_1$.
2. $F_1(\cdot) : \mathbb{T} \rightarrow g$ is small and $\sigma(F_1) = \{k_1\}$.
3. $\phi : \mathbb{T} \times G \rightarrow \mathbb{C}$ is spectrally supported in an obstruction of a given bidegree $(m-p, p)$, and $m-2p = b$: $\phi = Ob_1\phi = \sum_{k+b\frac{k_0}{2}=0, 0 \leq j \leq m} \phi_{k,j,p}^m e^{2i\pi kx} \pi_m^{j,p}$, and $m < N_1$

4. $F'_1(\cdot)$ is quadratic with respect to $F_1(\cdot)$
5. $\sigma(\text{Ad}(B_1(\cdot)).F'_1)$ is outside some ball of radius $N_2 \gg N_1$, where $B_1(\cdot)$ is the conjugation reducing $(\alpha, A_1 e^{F_1(\cdot)})$ to (α, A_2) where $A_2 = \{e^{2i\pi\epsilon/2}.e^{\{0, \hat{F}_1(k_1)\}}, 0\}$
6. The cocycle

$$\text{Conj}_{B_1(\cdot)}(\alpha, A e^{F_1(\cdot)} e^{F'_1(\cdot)}) = (\alpha, A_2 e^{F_2(\cdot)} e^{F'_2(\cdot)})$$

satisfies conditions in items 1 to 5, with N_2, N_3 in the roles of N_1, N_2 .

7. The angle between Ob_2 and Ob_3 in $(\mathcal{P}_m)^*$ is close to a root of the corresponding Legendre polynomial $p_{m/2}$ (cf. lemma 2.2):

$$\|Ob_3 \circ Ob_2\|_{L^2} \leq C N_2^{-s_0} \quad (11)$$

for some constants $C > 0$ and $s_0 \gg 0$.

We define the obstruction, truncation and rest operators for (α, A_i) as in the previous section. Then,

1. $\widetilde{Ob}_1\phi(\{z, w\}) = Ob_1\phi \circ (0, B_1(\cdot)) = \sum_{k+b\frac{k_1}{2}=0} \phi_{k,j,p}^m \pi_m^{j,p}$ is constant in x
2. We have

$$\|(Ob_2(\widetilde{Ob}_1\phi))\|_{L^2} \leq \|\phi\|_{L^2}$$

3. $(Ob_2(\widetilde{Ob}_1\phi)) \circ (0, B_1^*(\cdot))$ is spectrally supported in the image of Ob_1
4. If we call $\widetilde{Ob}_2\phi = (Ob_2(\widetilde{Ob}_1\phi)) \circ (0, B_2(\cdot))$, then $\widetilde{Ob}_3 = Ob_3(\widetilde{Ob}_2\phi) \circ (0, B_1^*.B_2^*(\cdot))$ satisfies

$$\|\widetilde{Ob}_3\|_s \leq C.C_{m,s} N_2^{-s_0+\lambda} \|\phi\|_{L^2}, \quad 0 \leq s \leq s_0 - \lambda$$

where C is the constant in eq. 11, $C_s > 0$ depends only on s , and $\lambda > 0$ is a universal constant.

Therefore, we can eliminate the obstructions activated at a certain step of the K.A.M. scheme, up to a small error, if we wait long enough so that the projections corresponding to two successive constants form an angle close to a root of $p_{m/2}$. The lemma indicates the following condition under which DUE should hold.

Condition 1. The condition for DUE (which we will show to be generic) is that for any $m \in \mathbb{N}^*$,

$$|\cos^2(\zeta_i) - \xi_{m/2}| = O(N_{n_i}^{-\infty})$$

along a subsequence of $\{n_i\}$, and where $p_{m/2}(\xi_{m/2}) = 0$, where $\zeta_i = \frac{1}{2} \arctan \left| \frac{\hat{F}_{n_i}(k_{n_i})}{\epsilon_{n_i}} \right|$. An equivalent way to state the condition is

$$\limsup \frac{-\log d(\cos^2(\zeta_i), \Xi_m)}{\log N_{n_i}} = +\infty, \quad \forall m \in \mathbb{N}^*$$

where $\Xi_m = \{\xi_m^j\}_j$ are the roots of the corresponding Legendre polynomial.

Proof of lemma 7.6. In coordinates where $A_1 = \{e^{2i\pi(k_1\alpha+\epsilon_1)/2}, 0\}$ is diagonal, the obstructions to the solution of the cohomological eq. over (α, A_1) are the modes $e^{2i\pi k_{m,p}x} \pi_{j,p}^m$ such that

$$k_{m,p} + (m - 2p) \frac{k_1}{2} = 0 \quad (12)$$

and the operator $Ob_1 = Ob_N^{k_1}$ is precisely the projection on these modes, which by hypothesis are also the spectral support of ϕ . This implies that the equation

$$\psi(x + \alpha, e^{-F_1(x)} A_1^* \cdot \{z, w\}) - \psi(x, \{z, w\}) = \phi(x, \{z, w\})$$

cannot be solved with good estimates.

In the same coordinate system, the conjugation $B_1(\cdot)$ is of the form

$$\begin{pmatrix} e^{-2i\pi \frac{k_1}{2} \cdot} & 0 \\ 0 & e^{2i\pi \frac{k_1}{2} \cdot} \end{pmatrix}$$

However, by hypothesis,

$$B_1(x + \alpha).A_1.e^{F_1(x)}.B_1^*(x) = \{e^{2i\pi\epsilon_1/2}, 0\}.e^{\{0, \hat{F}_1(k_1)\}} = A_2$$

By lemma 7.1, such a conjugation transforms the obstruction of the type $e^{2i\pi k_{m,p}x} \pi_m^{j,p}$ to the harmonic $\pi_m^{j,p}$, which proves the first point.

We remark that the vectors $z^l w^l$ are fixed by the reduction of resonant modes, and so is the corresponding line in the matrix of π_m , the harmonics $\pi_m^{j,l}$ with $l = m/2$. These are the only obstructions corresponding to $k = 0$ in the coordinates $(x, \{z, w\})$. Therefore, in the coordinates $(x, B_1(\cdot). \{z, w\})$ the rhs of the cohomological equation is written in the form

$$\begin{aligned} \tilde{\phi}(x, B_1(\cdot). \{z, w\}) &= \widetilde{Ob}_1 \phi(x, B_1(\cdot). \{z, w\}) = \sum_{j,p} \phi_{k_{m,p},j,p}^m \pi_m^{j,p}(z, \bar{z}, w, \bar{w}) \\ &= \sum_{j,p} \tilde{\phi}_{0,j,p}^m \pi_m^{j,p}(z, \bar{z}, w, \bar{w}) \end{aligned}$$

where the summation is over the resonant modes as defined in eq. 12. The obstruction is now constant in x .

If we denote by $(x, \{z_2, w_2\})$ the new coordinates in which A_2 is diagonal, the linearized equation assumes the form

$$\tilde{\psi}'(x + \alpha, A_2^* \cdot \{z_2, w_2\}) - \tilde{\psi}'(x, \{z_2, w_2\}) = \sum_{j,p} \tilde{\phi}_{0,j,p}^{m,(1)} \pi_m^{j,p}(z_2, \bar{z}_2, w_2, \bar{w}_2)$$

where a priori all harmonics in \mathcal{E}_{π_m} are authorized in the summation. As we have seen, the obstructions to this type of equation which are constant in x are the harmonics of the form $\pi_m^{j,m/2}$. A change of coordinates back to the system where A_1 is diagonal will give in general a full matrix $\pi_m^{j,p}$, $0 \leq j, p \leq m$. Then, composition with $(0, B_1^*(\cdot))$ implies the third point.

Let us now come to the fourth point. The obstructions $Ob_2(\widetilde{Ob}_1\phi)$ are constant in x , and remain so after conjugation by $(0, B_2(\cdot))$, since $B_2(\cdot)$ commutes with A_2 . Projection on Ob_3 gives a mapping satisfying $\|Ob_3(\widetilde{Ob}_2\phi)\|_{L^2} \leq CN^{-s_0}\|\phi\|_{L^2}$. Conjugation by $(0, B_1^*.B_2^*(\cdot))$ gives a mapping with (eventually) full support in harmonics, $\pi_m^{j,p}$, $0 \leq j, p \leq m$, and support in $[-m(N_1 + N_2), m(N_1 + N_2)]$ in the frequency space, which implies the fourth point, since $N_1 + N_2 \approx N_2$. \square

We can now return to the proof of the existence of an approximate solution under condition 1. Genericity is proved in the next section.

Proof of theorem 1.3. It is just an iteration of the previous lemma. The solution of the cohomological equation in step i of the K.A.M. scheme is done in the coordinates $(x, \{z_i, w_i\})$ and the change of coordinates $(x, \{z, w\}) \mapsto (x, \{z_i, w_i\})$ is given $\prod_1^{i-1} (0, D_i)(0, B_i(\cdot))$, where the $B_i(\cdot)$ are produced by the K.A.M scheme, and the D_i , defined in G/\mathcal{N}_i , diagonalizes A_i in the standard system of coordinates. Condition 2.2 implies that for any given $m = 2l \in \mathbb{N}$, there exists a subsequence $\{i_j\}$ along which point 11 of lemma 7.6 holds. This implies that the error term becomes arbitrarily small in the standard coordinates, and the proof is concluded. \square

8 Genericity of condition 1

We now prove the genericity of condition 1, thus concluding the proof of theorem 1.3. Genericity is obtained roughly in the same way as genericity of condition of eq. 8.

Let the cocycle $(\alpha, Ae^{F(\cdot)})$ be in normal form. By choosing i_0 big enough, the tail of perturbations corresponding to $n_i, i \geq i_0$ can be made arbitrarily small in C^∞ . Let us fix, for every $m > 0$, ξ_m a root of the Legendre polynomial of lemma 2.2 and a subsequence $i_l^{(m)}$ of indices $\geq i_0$. We can change (within the limitations of the K.A.M. scheme) the parameters $|\hat{F}_{i_l^{(m)}}(k_{i_l^{(m)}})|$ (for the given $k_{i_l^{(m)}}$) and $\epsilon_{i_l^{(m)}}$, so that condition 1 is satisfied exactly at the step $i_l^{(m)}$, and consequently condition 1 is dense. Since we only demand to be $O(N_{i_l^{(m)}}^{-\infty})$ -close to the root, the condition is also a G_δ by its very definition.

In particular, the elimination of the modes $\pi_2^{1,1}$ (the line in the matrix of π_2 corresponding to the polynomial zw) demands that $\epsilon_{n_{i_l^{(2)}}} = O(N_{i_l^{(2)}}^{-\infty})$ and $\hat{F}_i(k_i) \neq 0$, which implies a fortiori condition of eq. 8

9 Proof of theorem 1.6

In this section, we identify the cohomologically stable cocycles. They are in fact, in one case, reducible, and in the remaining one they are a particular case

of UE and not DUE cocycles, the cocycles corresponding to $\lim = \liminf = 0$ in formula 8.

Proof of theorem 1.6. Let the cocycle $(\alpha, Ae^{F(\cdot)})$ be in normal form (for a fixed choice of parameters), with an infinite number of steps $\{n_i\}$ where resonances are reduced, and call

$$\bar{\nu} = \limsup \nu_i = \limsup \frac{-\log |\varepsilon_i|}{\log N_{n_i}} \in [\nu, \infty]$$

If $\bar{\nu} < \infty$, then the K.A.M. scheme with $\nu' = 2\bar{\nu}$ sees only a finite number of resonances. Such a scheme can be applied in the following way. We start the scheme with the fixed parameter ν , and go deep enough so that the smallness conditions of the scheme with the new parameter ν' are satisfied. The new scheme produces a converging product of conjugations. This procedure amounts to reducing the initial perturbation to such a size that α can be placed in $DC(\gamma, \tau')$, for a convenient $\tau' > \tau$, and re-initiating the scheme for the rotation satisfying this weaker Diophantine condition (see also [Kar14] for a similar argument).

If, now, $\bar{\nu} = \infty$,⁴ an infinite subsequence of steps of the scheme, still denoted by $\{n_i\}$, reduces essential resonances, and we separate 2 cases. Either $\nu_i = \infty$ for all $i > i_0$ big enough, or $\nu_i < \infty$ along a subsequence.

In the first case, from the step i_0 on, all successive reductions of resonant modes form an angle of $\pi/2$. Then, it can be seen that the space of obstructions is the space generated by

$$\{\pi_m^{j,l}(z_{i_0}, \bar{z}_{i_0}, w_{i_0}, \bar{w}_{i_0}), m = 2l, m \in \mathbb{N}^*, 0 \leq j \leq m\}$$

(expressed in the standard coordinates), and that all modes orthogonal to obstructions are exact coboundaries. This is so, because the space of obstructions and the space orthogonal to it are preserved by the reduction of resonant modes. The small denominators are either 0 in the space of obstructions, or $> K_{n_i}$ in the orthogonal, so that in the space orthogonal to the obstructions, the sequence of approximate solutions converges, and the cocycle is cohomologically stable.

In the second case, the argument follows the classical proof of the fact that Liouvillean rotations are not cohomologically stable. Let us fix $m > 0$ and a subsequence such that $\nu_i < \infty$, but $\{\nu_i\} \rightarrow \infty$. At the step $i > i_0$, we choose $\phi'_i(x, \{z_i, w_i\}) = \phi'_i(\{z_i, w_i\}) \in \mathcal{E}_{\pi_m} \setminus \{0\}$, so that $\phi'_i = Ob_i \phi'_i$. Therefore, ϕ'_i is a coboundary over (α, A_i) , only with bad estimates for the solution. Let ψ'_i be the solution, so that $\|\psi'_i\|_{L^2} = N_{n_i}^{\nu_i} \|\phi'_i\|_{L^2}$. Then,

$$\phi_i = \phi'_i + (\alpha, A_i e^{F_i}) \cdot \psi'_i - (\alpha, A_i) \cdot \psi'_i$$

is a coboundary, and it is $N_{n_i}^{\nu_i+s+2} \|\phi'_i\|_{L^2} \|F_i\|_s$ close to ϕ'_i in C^s norm. We also choose ϕ'_i so that $\|Ob_i \phi_i\|_{L^2} \approx N_{n_i}^{-\nu_i}$. We now call $\tilde{\phi}_i$ the function ϕ_i expressed

⁴This is weaker than condition 1.

in the standard coordinates, and call

$$\tilde{\phi}(x, \{z, w\}) = \sum_{i=0}^{\infty} \tilde{\phi}_i(x, \{z, w\}) \in C^\infty(\mathbb{T} \times G)$$

The function $\tilde{\phi}$ is a coboundary, since it is a limit of coboundaries and this latter space is closed by hypothesis. Let $\tilde{\psi} \in C^\infty$ be the solution of the cohomological equation:

$$(\alpha, Ae^{F(\cdot)}) \cdot \tilde{\psi} - \tilde{\psi} = \tilde{\phi}$$

This equation implies that the shell of frequencies $[\frac{1}{2}N_{n_i}, \frac{3}{2}N_{n_i}]$ must carry an L^2 norm of the order of 1, which is incompatible with $\tilde{\psi} \in C^\infty$. \square

10 Necessity of the condition for DUE

In this section we let condition 1 degenerate in a controlled way in order to show that a condition of this kind is in deed necessary in order to obtain DUE. We prove the following proposition, which is to be compared with section 6.4.

Proposition 10.1. *Let $m \in \mathbb{N}$ be fixed, and $\Xi_m = \{\xi_m^j\}_j$ the roots of the corresponding Legendre polynomial. Let also $(\alpha, Ae^{F(\cdot)})$ be in normal form. Suppose, moreover, that*

$$\limsup \frac{-\log d(\cos^2(\zeta_i), \Xi_m)}{\log N_{n_i}} = s_0 < \infty$$

for some $\lambda < s_0 < \infty$ and

$$\limsup \frac{-\log d(\cos^2(\zeta_i), \Xi_{m'})}{\log N_{n_i}} = \infty, m' \neq m$$

Then, C^∞ coboundaries are dense in $C^{s_0-\lambda}$ but not in $C^{s_0+\lambda}$.

Proof. The proof that C^∞ coboundaries are dense in $C^{s_0-\lambda}$ is just a finite regularity version of the corresponding part of the proof of theorem 1.3, following lemma 7.6.

The proof that C^∞ coboundaries are not dense in $C^{s_0+\lambda}$ goes like the proof of theorem 1.6. \square

11 The local and global pictures of the dynamics

Using the theorems proved in this article and those that we cited in the introduction, we can give the following description of the phenomena encountered in the class of quasi-periodic cocycles over a diophantine rotation.

The local picture for DC rotations. Reducible cocycles form a dense but meager set in the neighborhood of constants, and reducible and cohomologically

stable cocycles form a meager set within this class (though for the natural topology). Reducible cocycles are accumulated by UE cocycles (by [Eli02]). UE cocycles are also generically not cohomologically stable, but there exist UE and cohomologically stable cocycles: they are the UE cocycles which are "far away" from reducible ones. Between UE and reducible cocycles one encounters C^∞ cocycles which are H^σ -reducible, for any fixed $\sigma \in [0, \infty)$.

In the regime of UE cocycles, a G_δ -dense subset satisfies the more restrictive property of being DUE. When one exits DUE within the UE regime, then invariant distributions of arbitrarily high orders can be created. The space of such distributions is always of infinite dimension. Finally, a given a DUE cocycle, a generic 0-mean-value function is not a coboundary (just as for Liouvillean rotations in tori).

The global picture for RDC rotations. The preceding picture holds true in an open-dense set of the total space. The rest of the space is filled by the immersed local Fréchet manifolds (of positive and finite codimension) formed by the cocycles that are conjugate to the normal forms $(\alpha, E_r(\cdot))$, $r \in \mathbb{Z}^*$, where $E_r(\cdot) = \{e^{2i\pi \cdot}, 0\}$. These cocycles, as already discussed, preserve a foliation with leaves $\mathbb{T} \times \mathbb{T}$, and in each leaf the dynamics is UE, but preserves countably many distributions in $H^{-1}(\mathbb{T} \times \mathbb{T})$. This is the content of theorem 1.7.

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